An Octagon of Oppositions in First Order Intuitionistic Logic with Strong Negation

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Abstract

In this paper I present a Hilbert style axiomatic system for intuitionistic first-order logic with strong negation. I show that in this system we have an octagon of opposition. This rests on the facts that (1) intuitionistic logic is a S4 modal logic and thus truth is persistent, i.e. truth means necessary truth and (2) necessity does not commute with universal quantification.

1 Introduction

It is a well known fact that there is no square of oppositions in standard intuitionistic logic (IL). The main formal and ineluctable reason is that there is no duality between A is proved (= is true) and the intuitionistic negation of A, noted $\neg A$ ($= A \to \bot$), is proved. This is based on the very fundamental property that A and $\neg\neg A$ are *not* logically equivalent in intuitionistic logic. This, in turn, is the consequence of the fact that intuitionistic truth is persistent - when A is proved, it is definitively proved, while not to be proved is not persistent: From A is not true we cannot conclude that $\neg A$ is, because if A is not proved it may be proved in a subsequent step of the development of the theory or $\neg A$ may be proved in a subsequent step of the development of the theory.

The question is: Is it possible to have a logic slightly stronger than IL with a negation connective, let say "$\sim$" such that $\sim\sim A$ is logically equivalent to A? The answer is yes and this follows from Nelson concept of "Constructive falsity". In few words, Nelson noticed that even though standard intuitionistic logic distinguishes (as it should) between $\neg(\forall x)\neg A(x)$ and $(\exists x)A(x)$, it should also be able to express, in number theory, that $\neg\forall x A(x)$ is true only if there is a proof of $\neg A(n)$ for some n. The problem is that $\neg(\forall x)A(x)$ is true whenever there is a constructive proof of $(\forall x)A(x) \to \bot$, proof that does not, in general, give such a n.

In a previous paper (Lepage 2016), I showed that in propositional intuitionistic logic with strong negation there is a square of opposition. In the present paper, I will firstly briefly recall this proof. I will then look at what happen if we introduce strong negation in first order intuitionistic logic.

2 The Language L of $ILSN$

2.1 The Language

The language L of Intuitionistic Logic with Strong Negation is the standard language of classical propositional calculus. The sole differences will be the interpretation of negation and of implication. As usual, the set of atomic propositions is $AT = \{\top, p_0, ..., p_i, ...\}$ where $\top$ is a symbol that will denote the truth. The set LT of literals is the union of AT and set $\{\sim p_i | p_i \in AT\}$.

Definition .1. (Well formed formulas) The set of well formed formulas (WFF) is the smallest set such that
1. $AT \subseteq WFF$;
2. If $A, B \in WFF$ then $\sim A, (A \wedge B), (A \vee B), (A \rightarrow B) \in WFF$.

We define $\bot =_{def} \sim \top$.

2.2 An Axiomatic System

Definition .2. Let us consider the following Hilbert-style axiomatic system.

A1 $A \rightarrow (B \rightarrow A)$
A2 $(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$
A3 $(A \wedge B) \rightarrow A$
A4 $(A \wedge B) \rightarrow B$
A5 $A \rightarrow (A \vee B)$
A6 $B \rightarrow (A \vee B)$
A7 $(A \rightarrow C) \rightarrow ((B \rightarrow C) \rightarrow ((A \vee B) \rightarrow C))$
A8 $\bot \rightarrow A$
A9 $\sim\sim A \rightarrow A$
A10 $A \rightarrow \sim\sim A$
A11 $\sim (A \wedge B) \rightarrow (\sim A \vee \sim B)$
A12 $\sim (A \vee B) \rightarrow (\sim A \wedge \sim B)$
A13 $(A \wedge \sim A) \rightarrow \bot$
A14 $\sim (A \rightarrow B) \rightarrow (A \wedge \sim B)$
A15 $(A \wedge \sim B) \rightarrow \sim (A \rightarrow B)$
A16 $\sim A \rightarrow \ (A \rightarrow B)$
A17 $A \rightarrow (B \rightarrow (A \wedge B))$
A18 $(\sim A \vee \sim B) \rightarrow \sim (A \wedge B)$
A19 $(\sim A \wedge \sim B) \rightarrow \sim (A \vee B)$
A20 $\top$

The only rule is modus ponens.

Theoremhood and the consequence relation are defined as usual. We write $\Gamma \vdash A$ when A is a consequence of a set of sentences Γ. We call this system *intuitionistic logic with strong negation* ($ILSN$).

One can ask the following question: Is the introduction of strong negation trivializes the logic i.e., reduces the logic to the classical one? The answer is no, because we still don't have $A \vee \sim A$ nor $A \vee \neg A$.

2.3 A Kripke Semantics for $ILSN$

We define the notion of a (Kripke) frame and the notion of a canonical (Kripke) frame. The fundamental notion is that of *deductively close, saturated, consistent set* (DCSC), which will play a role similar to maximally consistent sets in classical logic.

Definition .3. A Kripke frame is a pair $\{W, R\}$ such that W is a set of nodes and R is a transitive and reflexive relation on W.

Definition .4. A Kripke model is a pair $<< W, R >, f >$ where $W, R\}$ is a Kripke frame and $f : LT \to \wp(A)$ is such that, for any p_i and any f, $f(p_i) \cap f(\sim p_i) = \varnothing$.

Intuitively, $w \in f(p_i)$ (resp. $w \in f(\sim p_i)$) means that p_i holds at node w (resp $\sim p_i$ holds at w). These conditions cannot hold together in any w even if they can be both not true in some w.

Definition .5. A set Γ of wff of $ILSN$ is a $DCSC$ iff

1. $A \in \Gamma$ iff $\Gamma \vdash A$;
2. $(A \vee B \in \Gamma$ iff $A \in \Gamma$ or $B \in \Gamma$;
3. $\Gamma \nvdash \bot$.

We can easily show that :

If Δ is a consistent set of wff of L and $\Delta \nvdash A$, then there is a $DCSC$ Γ such that $\Delta \subseteq \Gamma$ and $\Gamma \nvdash A$.

Definition .6. Let Γ be a $DCSC$. We call *bicharasteristic function* of Γ the partial function f_Γ such that $f_\Gamma : WFF \to \{0, 1\}$with

$f_\Gamma(A) = 1$ if $A \in \Gamma$;

$f_\Gamma(A) = 0$ if $\sim A \in \Gamma$;

$f_\Gamma(A)$ is undefined otherwise.

Definition .7. A *valuation* is a partial function $f : AT \rightarrow \{0.1\}$

To each valuation f correspond one and only one $\Gamma \in DCSC$. We note the extension of the valuation f'_Γ which is such that $f'_\Gamma(A) = 1$ iff $A \in \Gamma$.

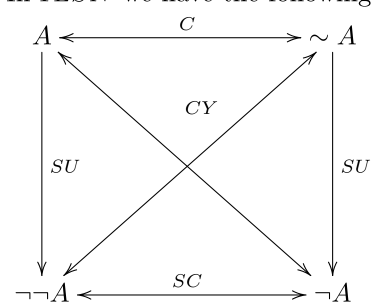
Let W be the set of all $DCSC$. Clearly, $\subseteq$ is a partial order on W. So, $< W, \subseteq >$ is a Kripke frame. Let f_Γ be any valuation.

$<< W, \subseteq >, f_\Gamma >$ is a model such that, for any wff A, $f'_\Gamma(A) = 1$ iff $A \in \Gamma$.

$< W, \subseteq >$ is a canonical Kripke frame, that is, for any A such that $\Delta \nvdash A$, there is a DCSC Γ with $\Delta \subseteq \Gamma$ such that $f'_\Gamma(A) \neq 1$.

2.4 A Square of Opposition

In $ILSN$ we have the following square of oppositions.



Abbreviations
C (Contrary)
SC (Subcontrary)
SU (Subaltern)
CY (Contradictory)

3 Adding quantifiers

3.1 The language L'

We first add to the language L of propositional calculus (with the two negations) a new symbol $\forall$, a denumerable set of constants $Con = \{a_0, ..., a_n, ...\}$, a denumerable set of variables $Var = \{x_0, ..., x_n, ...\}$ and a denumerable set of predicates $Pred = \{P^1_0, ..., P^1_n, ..., P^m_0, ..., P^m_n, ...\}$. It is convenient to adopt the convention that p_n is P^0_n. In P^m_n, m is the number of place of the predicate and n is a part of the name of the predicate (we call this language L'). With the above convention, an atom p_n is a 0-place predicate. The set T of terms is

$Con \cup Var$.

We have to introduce the notion of *basic well formed formulas* which will play a role similar to literals in the propositional calculus. The reason is that the logic is no more binary: the introduction of strong negation has the consequence that to be false is no more not to be true. There is a third possibility, to be undefined. The introduction of strong negation re-establish a duality between to be true and to be (strongly) false.

Definition .8. The set of basic well formed formulas ($BWFF_{L'}$) is the smallest set such that, for any $i, j \in \mathbb{N}$, $P^i_j(b_1, ..., b_k)$ and $\sim P^i_j(b_1, ..., b_k)$ where the b_k, $k \leq i$ are terms.

Definition .9. (Well formed formulas). (We drop the index L'). The set of well formed formula (WFF) is the smallest set such that :

1. $BWFF \subseteq WFF$;
2. If $A, B \in WFF$, then $\sim A, (A \wedge B), (A \vee B), (A \rightarrow B), (\forall (x_i) A) \in WFF$.

In the formula $(\forall (x_i) A)$, A is call the scope of $(\forall x_i)$. A variable x_i having an occurrence in A is said to be free in A if it is not in the scope of $(\forall x_i)$, otherwise it is said to be bound.

3.2 An Axiomatic System

Definition .10. An axiomatic system for first order intuitionistic logic ($AFOIL$) is provided :

A1 $A \rightarrow (B \rightarrow A)$
A2 $(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$
A3 $(A \wedge B) \rightarrow A$
A4 $(A \wedge B) \rightarrow B$
A5 $A \rightarrow (A \vee B)$
A6 $B \rightarrow (A \vee B)$
A7 $(A \rightarrow C) \rightarrow ((B \rightarrow C) \rightarrow ((A \vee B) \rightarrow C))$
A8 $\perp \rightarrow A$
A9 $\sim\sim A \rightarrow A$
A10 $A \rightarrow \sim\sim A$
A11 $\sim (A \wedge B) \rightarrow (\sim A \vee \sim B)$
A12 $\sim (A \vee B) \rightarrow (\sim A \wedge \sim B)$
A13 $(A \wedge \sim A) \rightarrow \perp$
A14 $\sim (A \rightarrow B) \rightarrow (A \wedge \sim B)$
A15 $(A \wedge \sim B) \rightarrow \sim (A \rightarrow B)$

A16 $\sim A \rightarrow\ (A \rightarrow B)$
A17 $A \rightarrow (B \rightarrow (A \wedge B))$
A18 $(\sim A \vee \sim B) \rightarrow \sim (A \wedge B)$
A19 $(\sim A \wedge \sim B) \rightarrow \sim (A \vee B)$
A20 $\top$
A21 $\forall(x_i)A(x_i) \rightarrow A(a_j|x_i)$ for any a_j
A22 $A(a_j) \rightarrow\sim \forall(x_i) \sim A(x_i|a_j)$ for any a_j
A23 $\sim \forall(x_i)A(x_i) \rightarrow\sim A(a_j|x_i)$ for some a_j

Abbreviation $(\exists x_i)A(x_i) =_{def}\sim (\forall x_i) \sim A(x_i)$

Besides modus ponens, we have two new rules:

$(\forall)gen$: If $\Gamma \vdash A \rightarrow B$, then $\vdash A \rightarrow (\forall x_i)B$ for x_i not free in A.
$(\exists)gen$: If $\Gamma \vdash (\exists x_i)A \rightarrow B$, then $\vdash A \rightarrow B$ for x_i not free in B.

The définitions of a theorem and of a derivation are the standard ones.
Definition .11. Let Γ be a set of wff of $WFF_{L'}$ such that,

1. $A \in \Gamma$ iff $\Gamma \vdash A$;
2. $(A \vee B) \in \Gamma$ iff $A \in \Gamma$ or $B \in \Gamma$;
3. $\Gamma \nvdash \bot$.

We say that Γ is a deductively close, saturated and consistent set $(DCSC)$ of *wff*.

Notice that this definition is the same as for the propositional case excepted that *WFF* and $\vdash$ are, of course, not the same.

The fundamental property of the $(DCSC)$ is given by the following proposition.
Proposition .1. Let Γ be a consistent set of *wff* and $A \nvdash \Gamma$, then there is a $DCSC$ Δ such that $\Gamma \subseteq \Delta$ and $A \notin \Delta$.

Proof

(This proof is not constructive.)
A is called the test formula. Let $E =< E_0, E_1, E_2, ... >$ be an enumeration of all *wff*s where each *wff* appears denumerably many times. We define the following sequence of sets:

$\Gamma_0 = \Gamma$;

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$\Gamma_{k+1} = \Gamma_k$ if $\Gamma_k \cup \{E_k\} \vdash A$;
$\Gamma_{k+1} = \Gamma_k \cup \{E_k\}$ if $\{\Gamma_k\} \vdash E_k$, and E_k is not $(B \vee C)$;
if E_k is $(B \vee C)$, $\Gamma_{k+1} = \Gamma_k \cup \{E_k\} \cup \{B\}$ if $\Gamma_k \cup \{E_k\} \cup \{B\} \nvdash A$
else $\Gamma_{k+1} = \Gamma_k \cup \{E_k\} \cup \{C\}$.

We define

$$\Delta = \bigcup_{k=0}^{\infty} \Gamma_k$$

Claim

(1) $\Delta \nvdash A$

We first show that, for any k, $\Gamma_k \nvdash A$.

For $k = 0$, it is trivial. Let us suppose that $\Gamma_k \nvdash A$, we show that $\Gamma_{k+1} \nvdash A$.

If $\Gamma_{k+1} = \Gamma_k \cup \{E_k\}$ because $\Gamma_k \vdash E_k$, and E_k is not $(B \vee C)$, we easily get the result.

Let us suppose that E_k is $(B \vee C)$.

If $\Gamma_{k+1} = \Gamma_k \cup \{E_k\} \cup \{B\}$ because $\Gamma_{k+1} \nvdash A$, it is trivial;

If $\Gamma_{k+1} = \Gamma_k \cup \{E_k\} \cup \{C\}$ because $\Gamma_k \cup \{E_k\} \cup \{B\} \vdash A$, we have to show that $\Gamma_k \cup \{E_k\} \cup \{C\} \nvdash A$.

Let us suppose that $\Gamma_k \cup \{E_k\} \cup \{C\} \vdash A$.

From The axiom A7, the deduction theorem and the fact E_k is $(B \vee C)$ we have that

$\Gamma_k \cup \{(B \vee C)\} \vdash A$ which contradicts the hypothesis.

(2) If $\Delta \vdash B$, then $B \in \Delta$ because B is one of the E_k.

(3) Δ is saturated, i.e., if $B \vee C \in \Delta$, then $B \in \Delta$ or $C \in \Delta$. It is a trivial consequence of the definition of the Δ's.

(4) Δ is consistent. This follows from the fact that $\Delta \nvdash A$.

□

A very important consequence of the proposition.1 is the following: if W is a consistent set and A is classically valid such that $A \notin W$, there is a $DCSC$ Γ such that $W \subseteq \Gamma$ such that $A \notin \Gamma$.

We can now use $DCSC$'s for providing a semantics for $AFOIL$

3.3 A Kripke Semantics for $AFOIL$

Let us recall that a Kripke frame is a pair $< W, R >$ where W is a set of nodes and R is a reflexive and transitive relation on W. Let D be a denumerable set of objects. For any $n \in \mathbb{N}$ we consider the set of $\wp(D^n)$ of subsets of D^n.

Definition .12. (Interpretation)

An *interpretation* I for L' is a triplet $< D, (\ )_I^{Pred}, (\ )_I^{Con} >$ where:
1. D is a domain;
2. To each predicate letter P_m^n is assigned $(P_m^n)_I^{Pred}$ which is a set of n-uple of elements of D.
3. To each constant a_i is assigned $(a_i)_I^{Con}$ which is a fixed element of D.
The connectives receive (even $\sim$) there usual interpretation of Kleene strong connectives. The interpretation of $\rightarrow$ is the usual interpretation in intuitionistic logic.

We can now define a Canonical Kripke frame.

Proposition .2. The pair $< DCSC, \subseteq >$ is a canonical Kripke frame. The canonical model in the canonical Kripke frame is the 3-uple $<< DCSC, \subseteq >, \vdash >$. We have to remember that for any $\Gamma \in DCSC$, $\Gamma \vdash A$ iff $A \in \Gamma$.

Proof

The fact that $< DCSC, \subseteq >$ is a Kripke frame is trivial. The fact that $<< DCSC, \subseteq >, \vdash >$ is a model is also trivial. The model is a canonical one

because if Δ is a consistent set such that $A \notin \Delta$, there is a $DCSC$ Γ, $\Delta \subseteq \Gamma$, such that $A \notin \Gamma$ and thus $\Gamma \nvdash A$.
□

Proposition .3. Any $DCSC$ Γ define a 3-value valuation.

Proof Let A be any wff, and let f_Γ be such that:
$f_\Gamma : WFF \rightarrow \{0, 1, u\}$ $f_\Gamma(A) = 1$ iff $A \in \Gamma$
$f_\Gamma(A) = 0$ iff $\sim A \in \Gamma$
$f_\Gamma(A) = u$ (undefined) otherwise.
We have $f_\Gamma(\sim A) = 1$ iff $\sim A \in \Gamma$ and
$f_\Gamma(\sim A) = 0$ iff $\sim\sim A \in \Gamma$ iff $A \in \Gamma$
Otherwise, $f_\Gamma(A) = f_\Gamma(\sim A) = u$.

□

Proposition .4. Let A be any wff and Γ any $DCSC$. There is three and only three possibilities :

1. $A \in \Gamma$ and $\sim A \notin \Gamma$
2. $\sim A \in \Gamma$ and $A \notin \Gamma$
3. $A \notin \Gamma$ and $\sim A \notin \Gamma$

If A is classically valid, only 1. and 3. are possible options.
If A is a classical contradiction, only 2. and 3. are possible options.

Proof

This is a straightforward consequence of proposition .3.

□

We easily prove soundness and completeness but the proofs are too longer to be given here. As these proofs are simple generalization of those of the propositional case, we once more refer the reader to Lepage (2016). The proofs go along the following lines. For soundness, we check directly that every axiom is valid for the interpretation given above and we check that the rules transmit validity. For completeness, we suppose that some valid wff A is not a theorem and then, using what we called the fundamental property, that there is a member of $DCSC$ that does not contain A so A is not true at this node in the canonical model.

Proposition .5. Let us consider the following four sets of pairs of wff.

$P_C = \{< (\forall x)A, (\forall x) \sim A >, < (\forall x)\neg\neg A, (\exists x) \sim A >, < (\exists x)A, (\forall x)\neg A >\}$
$P_{SC} = \{< (\exists x)\neg\neg A, (\exists x)\neg A >, < (\exists x)A, (\exists x) \sim A >, < (\forall x)\neg\neg A, (\forall x)\neg A >\}$
$P_{SU} = \{< (\forall x)A, (\exists x)A >, < (\exists x)A, (\exists x)\neg\neg A >, < (\forall x)A, (\forall x)\neg\neg A >,$
$< (\forall x)\neg\neg A, (\exists x)\neg\neg A >, < (\forall x) \sim A, (\exists x) \sim A >, < (\exists x) \sim A, (\exists x)\neg A >,$
$< (\forall x) \sim A, (\forall x)\neg A >, < (\forall x)\neg A, (\exists x)\neg A >\}$
$P_{SY} = \{< (\forall x)A, (\exists x)\neg A >, < (\forall x) \sim A, (\exists x)\neg\neg A >\}$
The 3 pairs of P_C are contrary pairs.
The 3 pairs of P_{SC} are sub-contrary pairs.
The 8 pairs of P_{SU} are subaltern pairs (+ 2 by transitivity).
The 2 pairs of P_{SY} are contradictory pairs.

Proof

We give a proof for four cases, the others are left to the reader. For P_C, let us suppose that $(\forall x)A, (\forall x) \sim A$ are both true in some Γ. Applying A21 to both, we have that $A(a|x)$ and $\sim A(a|x)$ are in Γ and thus Γ is inconsistent and is not a $DCSC$.

Thus, they can both be false. Let us suppose that we have $\sim (\forall x)A$ and $\sim (\forall x) \sim A$. Using A23 twice, we get $\sim A(a_i|x)$ for some a_i and $\sim\sim A(a_j|x)$ for some a_j. By A9, we get $A(a_j|x)$, and both can be true at the same time.

The second pair is $< (\forall x)\neg\neg A, (\exists x) \sim A >$ i.e., $< (\forall x)\neg\neg A, (\sim \forall x) \sim\sim A >$. Let us suppose they are both in the same Γ. By A21, we get $\neg\neg A(a_i|x)$ for any a_i. From the second term we get, $\sim\sim\sim A(a_j|x)$ for some a_j. Using A9, we get $\sim A(a_j)$. But $\sim A(a_j) \to \neg A(a_j)$ i.e., $\sim A(a_j) \to (A(a_j) \to \bot)$ by A16. Taking $i = j$, we have both $\neg\neg A(a_j)$ and $\neg A(a_j)$ i.e., $((A(a_j) \to \bot) \to \bot)$ and $(A(a_j) \to \bot)$ which leads to a contradiction.

Thus, they can both be false i.e., $\sim (\forall x)\neg\neg A$ and $\sim (\forall x) \sim A$.

From the first member, we get $\sim \neg\neg A(a_i)$ for some a_i i.e., $\sim ((A(a_i) \to \bot) \to \bot)$. By A14, we get $((A(a_i) \to \bot) \wedge \sim \bot)$ and finally $\neg A(a_i)$. From the second member $\sim\sim\sim A(a_j)$ for some a_j. By A10 we get $\sim A(a_j)$. But $\neg A(a_i)$ and $\sim A(a_j)$ can be both true and we have the expected result.

Let us consider a pair in P_{SC}
We prove that $(\exists x)\neg\neg A$ and $(\exists x)\neg A$ can be both true but they can't be both false. If $(\exists x)\neg\neg A$ hold in Γ, then $\sim (\forall x) \sim \neg\neg A$ and $\sim\sim \neg\neg A(a_i)$ holds in Γ for some a_i. Furthermore, by A9, $\neg\neg A(a_i)$. We apply the same treatment to $(\exists x)\neg A$ and we show that $\neg A(a_j)$ hold in Γ for some $A(a_j)$. This is consistent.

We show that they can't be both false. Let us suppose that $\sim(\exists x)\neg\neg A$ is in Γ. Then $\sim\sim(\forall x)\sim\neg\neg A$ is also in Γ by A9. Then $(\forall x)\sim\neg\neg A$ is also in Γ. This implies that $\sim\neg\neg A(a_i)$ is in Γ for all a_i. This means that $\sim(\neg A(a_i)\rightarrow\bot)$ is in Γ. By A14, $(\neg A(a_i)\wedge\sim\bot)$ is in Γ. Thus $\neg A(a_i)$ is in Γ.

We apply the same treatment to $\sim(\exists x)\neg A$ and we get that for any a_j, $A(a_j)$ is in Γ. We reach a contradiction.

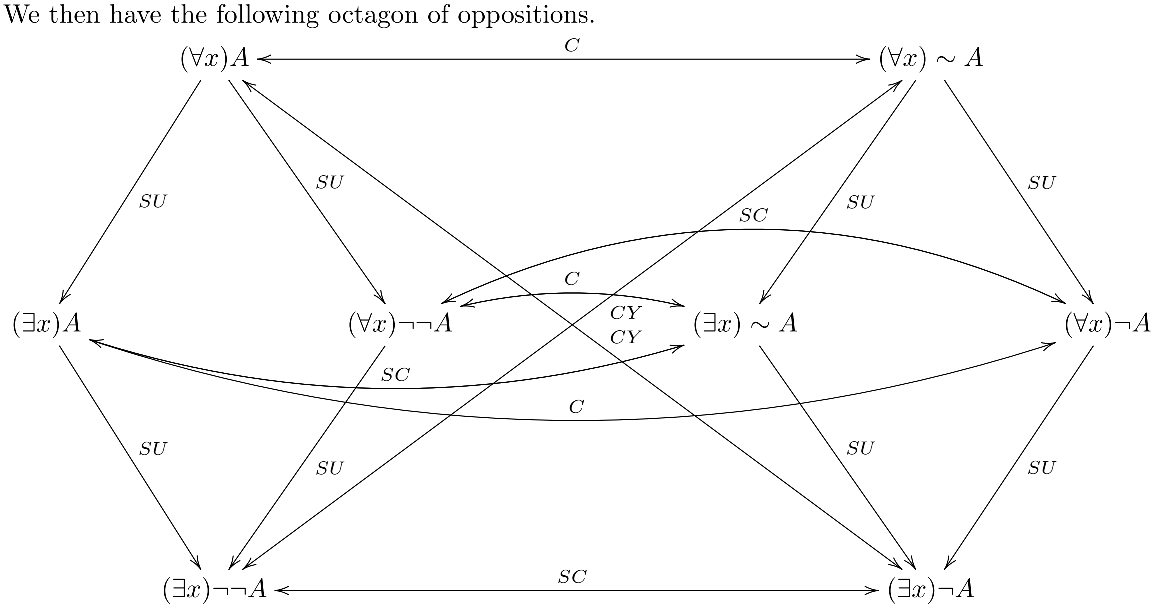
Our fourth example is for a pair in P_{SU}.

Let us consider the pair $<(\exists x)A, (\exists x)\neg\neg A>$. We show that $(\exists x)A$ and $\sim(\exists x)\neg\neg A$ lead to a contradiction.

From $(\exists x)A$ we get $\sim(\forall x)\sim A$. We derive $\sim\sim A(a_i)$ for some a_i and thus $A(a_i)$.

From $\sim(\exists x)\neg\neg A$ we get $\sim\sim(\forall x)\sim\neg\neg A$. By A9, we get $(\forall x)\sim\neg\neg A$. By S21, we get $\sim\neg\neg A(a_j)$ for all $A(a_j)$. This is an abbreviation for $\sim((A(a_j)\rightarrow\bot)\rightarrow\bot)$. By A14, we derive $((A(a_j)\rightarrow\bot)\wedge\sim\bot)$ and then we derive $(A(a_j)\rightarrow\bot)$.

Finally, taken $j=i$ we get $A(a_i)$ and $(A(a_i)\rightarrow\bot)$ and using MP we have a contradiction.
□
We then have the following octagon of oppositions.



The labels of arrows have the same meaning as for the square.

4 Conclusion

Rather than state a conclusion, I will suggest a moral to this story. The natural aspect and the elegance of this construction shows, if it were still necessary, that it is possible to restore the duality between truth and falsity in intuitionistic logic without affecting its fundamental concept and principle which are the rejection of the law of excluded middle and that of bivalence.

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