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# Categories of first-order quantifiers

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**Abstract.** One well known problem regarding quantifiers, in particular the $1^{st}$-order quantifiers, is connected with their syntactic categories and denotations. The unsatisfactory efforts to establish the syntactic and ontological categories of quantifiers in formalized first-order languages can be solved by means of the so called *principle of categorial compatibility* formulated by Roman Suszko, referring to some innovative ideas of Gottlob Frege and visible in syntactic and semantic compatibility of language expressions. In the paper the principle is introduced for *categorial languages* generated by the Ajdukiewicz's classical categorial grammar. The $1^{st}$-order quantifiers are typically ambiguous. Every $1^{st}$-order quantifier of the type $k > 0$ is treated as a two-argument functor-function defined on the variable standing at this quantifier and its scope (the sentential function with exactly $k$ free variables, including the variable bound by this quantifier); a binary function defined on denotations of its two arguments is its denotation. Denotations of sentential functions, and hence also quantifiers, are defined separately in Fregean and in situational semantics. They belong to the ontological categories that correspond to the syntactic categories of these sentential functions and the considered quantifiers. The main result of the paper is a solution of the problem of categories of the $1^{st}$-order quantifiers based on the principle of categorial compatibility.

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## 1. Introduction

Sentential logic beginnings date back to ancient times, times the Stoics ($3^{rd}$ c BC), while the notion of a quantifier arose only in XIX century and of the calculus of quantifiers, predicate logic – only at the beginnings of the XX century. The beginnings of this logic, as we know, were created by Gottlob Frege (Begriffschrift, [11]).

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The main problems and their solutions connected with the subject of this paper were presented at the Logic Colloquium'15 held in Helsinki on August 3-8, 2015, co-located with the $15^{th}$ International Congress of Logic, Methodology and Philosophy of Science, CLMPS (see my abstract [35]; see also [36]).

Around 1879 Frege and – independently – Charles Sanders Peirce developed a way to extend sentential logic by introducing symbols representing *determiners*, such as 'all', 'some', 'no', 'every', 'any', and so on.

Frege and Peirce used two symbols: the *universal quantifier* (which we will write $\forall$) corresponding roughly to the English 'all', 'every' and 'each' and the *existential quantifier* (which we will write $\exists$) corresponding to the English 'some', 'a', 'an'.

In this paper we will consider only standard, Fregean quantifiers $\forall$ and $\exists$ of the $1^{\text{st}}$-order as individual variable-binding operators. They are used in formulas of predicate logic of the $1^{\text{st}}$-order and in formalized languages of elementary theories based on this logic. Their syntactic role and semantic references, i.e. denotation, extension, created some problems that have not been satisfactorily solved yet.

In the next part (Section 2) I shall partially explicate the problem of quantifiers. In Section 3, I'll outline some intuitive foundations of my theory of categorial languages which gives the formal direction for justification of my solution of the problem of quantifiers. It corresponds to the principle $(CC)$ of categorial compatibility based on some Frege's ideas and was formulated by Roman Suszko [27]. The solution of the problem is presented in Section 4.

## 2. Problem of quantifiers

The problem of quantifiers is connected with the difficulty pertaining to establishing their syntactic and semantic categories.

Leśniewski's theory of semantic/syntactic categories [16, 17], which was improved by Ajdukiewicz [1] by introducing categorial indices, does not, obviously, solve this problem, which limits the universal character of the theory.

Leśniewski's hierarchy of semantic/syntactic category does not include any variable-binding operators. Leśniewski, in his protothetics and ontology systems, allows only one operator – the universal quantifier, noting it as parentheses, Ajdukiewicz, on the other hand, indicates the difficulty of assigning to quantifiers the index $s/s$ or $s/ns$.

Assigning to them the index $s/s$, i.e. the category of sentence-forming functors of one-sentence argument, would mean that the quantifiers belong to the same category as one-argument connectives, and assigning to them the index $s/ns$ of sentence-forming functors of one-name and one-sentence arguments would mean that we include them into the same category as some expressions of indirect speech, e.g. 'think that', 'know that', etc.

It has been suggested that the categorial grammar, which Bar-Hillel derived from Ajdukiewicz's version of the theory of semantic/syntactic categories, does not satisfactorily account for the role of bound variables and operators binding them.

Suszko [25, 27] assigns to them the index $s//s/n$, and thus the index of sentence-forming functor of the argument, which is a one-argument predicate. In this way, the index, for example in the sentence '$\forall x(x$ *flows*)' pertains to the entire quantifier-variable pattern '$\forall x(x \ldots)$' (see Simons [23]) which corresponds to English word 'everything' (see also Cresswell [7], Simons [24]).

**Suszko and many other researchers of language syntax treat quantifiers as expressions independent of the quantifier variable. Generally, researchers avoid bound variables in attempting to solve the problem**, for example by means of combinators (Curry [8, 9], Curry and Feys [10], see e.g. Simons [23]).

But earlier, Suszko stated that mounting variable-binding operators into a syntactic scheme requires general principles other than the theory of syntactic/semantic categories.

The principle $(CC)$ of categorial compatibility is one such principle. It allows us to assign to every expression of a formalized $1^{\text{st}}$-order language, which possesses an index symbolizing a syntactic category, a denotation whose ontological category (relative to the universe $U$ of a given model of the language) is indicated by the same index.

Suszko assumes that

- the denotation of the entire expression $\forall x(e(x))$, where $e(x)$ is a sentential function with the free variable $x$, is ether the logical value 1 (of truth) or the logical value 0 (of falsity) which belong to the ontological category with the index $s$, and
- the denotation of the universal quantifier $\forall$ is the function of generalization which has the value 1 in only one case, if its argument is the universe $U$.

The function of generalization belongs to the ontological category with the index $s//s/n$ because its arguments are any sets belonging to the family $P(U)$ included into the ontological categories with the index $s/n$. In this way the principle $(CC)$ holds although the principle of syntactic connection $(SC)$ does not hold because no index is assigned to quantifier variable $x$, and the scope of the quantifier $\forall$ (here $e(x)$) is not one-argument predicate of the syntactic category with the index $s/n$.

In the next parts of this paper I explicate both the principle $(SC)$ of syntactic connection and the principle $(CC)$ of categorial compatibility on the basis of my theory of categorial languages [30–34] which allows us to give some solutions to the problem of quantifiers.

The essence of the approach proposed here is considering them to be typical syntactic notions: functors-functions mapping language expressions into language expressions that correspond to some functions on extralinguistic objects – on denotations of arguments of these functors.

Let us note that a standard background for research in the field of mentioned quantifiers assumes treating them as some functions or relations on extralinguistic objects, mostly functions with index $t//t/e$ (cf. Mostowski [21], Lindström [18], Montague [19, 20], Nowaczyk [22], van Benthem [3, 4], van Benthem and Westerståhl [5]).

## 3. Some intuitive foundations of the theory of categorial languages

### 3.1. Main ideas of formalization of categorial language

In the paper, formal-logical considerations relate to syntax and extensional semantics of any language $L$ characterized *categorially*:

- in the spirit of some ideas of Husserl [15] and Leśniewski–Ajdukiewicz's theory of syntactic/semantic category (see Leśniewski [16, 17], Ajdukiewicz [1, 2]),
- in accordance with Frege's ontological canons [13],
- in accordance with Bocheński's motto [6]: *syntax mirrors ontology*, and
- some ideas of Suszko [25–28]: *language should be a linguistic scheme of ontological reality and simultaneously a tool of its cognition.*

The paper includes developing and some explications of these authors' ideas. It also presents, in a synthetic form, some ideas presented in my papers published in [30–34].

Language $L$ is there defined, if the set $S$ of all *well-formed expressions* (briefly *wfes*) is determined. These expressions must satisfy requirements of categorial syntax and categorial semantics.

### 3.2. Categorial syntax

The categorial syntax of $L$ is connected with generating the set $S$ by the classical categorial grammar and belonging *wfes* of $S$ to appropriate syntactic/semantic categories.

A characteristic feature of categorial syntax is that each composed *wfe* of the set $S$ has a *functor-argument* structure, in this sense that, in accordance with the principle originated by Frege [11], it is possible to distinguish in it its constituent called the *main functor*, and the other constituents — called *arguments* of that functor, yet each constituent of the *wfe* has a determined syntactic category.

If $e$ is a functor-argument *wfe* of $S$, $f$ is its main functor and $e_1, e_2, \ldots, e_n$ its subsequent arguments then $e$ can be written in the functional-argument form:

$$e = f(e_1, e_2, \ldots, e_n). \qquad (e)$$

In categorial approach to the language $L$, syntactic categories of *wfes* of $L$ are determined by attributing to them, like their expressions, categorial indices of a certain set $I$. To every *wfe* $e$ of the set $S$ is unambiguously assigned a categorial index (type) $i_S(e)$ of the set $I$; *wfes* belonging to the same syntactic category $CATa$ have the same categorial index $a$.

Categorial indices were introduced by Ajdukiewicz [1] into logical semiotics with the aim to determine the syntactic role of expressions and to examine their syntactic connection, in compliance with the principle of syntactic connection ($SC$) discussed below.

The set $S$ of all *wfes* of $L$ is then intuitively defined as the smallest set including the vocabulary of $L$ and closed with respect to the principle ($SC$), which in free formulation says that

($SC$) *The categorial index of the main functor of each functor-argument expression of the language $L$ is formed out of the categorial index of the expression which the functor forms together with its arguments, as well as out of the subsequent indices of arguments of this functor.*

In the formal definition of the set $S$ it is required that each functor-argument constituent of the given expression should satisfy the principle ($SC$).

If the functor-argument expression $e = f(e_1, e_2, \ldots, e_n)$ is a *wfe* (it belongs to the set $S$), then in accordance to the principle of syntactic connection ($SC$) the index of its main functor $f$ formed from the index $a$ of $e$ and successive indices $a_1, a_2, \ldots, a_n$ of successive arguments $e_1, e_2, \ldots, e_n$ of the functor $f$, can be written in the following quasi-fractional form:

$$i_S(f) = i_S(e)/i_S(e_1)i_S(e_2)\ldots i_S(e_n) = a/a_1a_2\ldots a_n. \qquad (\mathbf{i}_S)$$

**3.2.1. An algebraic structure of categorial language.** In categorial language $L$ we can distinguish two sets: the set $B$ of all basic *wfes* of $S$ and the set $F$ of all functors of $S$ such that

$$S = B \cup F \text{ and } B \cap F = \emptyset,$$

where functors of the set $F$ differ from basic expressions of $B$ that they have indices formed from simpler ones. If the functor $f$ has the functoral index of the form ($\mathbf{i}_S$), i.e. the index of the form $a/a_1a_2\ldots a_n$ then it belongs to the syntactic category $CATa/a_1a_2\ldots a_n$ and so to the category of functors forming expressions with the index $a$ if their arguments are $n$ expressions with successive indices $a_1, a_2, \ldots, a_n$. So the functor $f$ can be treated as the following partial function defined on *wfes* of $S$:

$$f : CATa_1 \times CATa_2 \times \ldots \times CATa_n \to CATa$$

mapping of *wfes* from Cartesian product of syntactic categories $CATa_1$, $CATa_2$, …, $CATa_n$ into the category $CATa$. Then we have

$$f \in CATa/a_1a_2\ldots a_n = CATa^{CATa_1 \times CATa_2 \times \ldots \times CATa_n}. \qquad (CAT_f)$$

In this way we simultaneously can regard the categorial language $L$ as an algebraic structure $\mathbf{L}$, partial algebra with the carrier $S$ and the set $Fo \subseteq F$ of partial functions on $S$ (simple functors of $L$):

$$\mathbf{L} = \langle S, Fo \rangle .$$

### 3.3. Categorial semantics

Categorial extensional semantics is connected with *denotations* of *wfes* of $S$ and with their belonging to an appropriate semantic extensional category. Each constituent of the composed *wfe* has determined a semantic extensional category and also a *denotation*, and thus – an *ontological category* (the *category of ontological objects*). *Denotations* (*extensions*) of *wfes* of $L$ are sets of *object references* (*references*) of *wfes* of $L$, objects of the cognized reality, e.g.: individuals, sets of individuals, states of affairs, operation on the indicated objects, and the like.

We will concentrate only on referential relationships between expressions of $L$ and reality to which they refer. We enrich the categorial grammar generating $L$ by the *denotation operation* $\delta$ regarded as its semantic component. The denotation operation $\delta$ assigns to every *wfe* of the set $S$ an object of ontological reality $ONT$ describing by the language $L$ – its denotation belonging to an ontological category. So

$$\delta : S \to ONT, \tag{$\delta$}$$

where $ONT$ is the sum of all ontological categories corresponding to *wfes* of $S$.

According to some innovative ideas of Frege [12, 13], Bocheński's (his famous motto: syntax mirrors ontology) and Suszko [25–27] who anticipated the research in categorial semantics and was the first to use categorial indices as a tool for co-ordination of expressions and their references, extralinguistic objects, the mutual dependence of syntactic and semantic formal description of $L$ should be considered by keeping the *principle* $(CC)$ *of categorial compatibility*, based on the compatibility of the syntactic category of each language expression of $L$ with the ontological category assigned to its denotation. The principle $(CC)$ of syntactic and semantic, i.e. also ontological categorial compatibility in Suszko's formulation can be given by keeping for any *wfe* $e$ of categorial language $L$ the relationship:

$$e \in CAT\iota \quad \text{iff} \quad \delta(e) \in ONT\iota, \tag{$CC$}$$

where $CAT\iota$ and $ONT\iota$ are: the syntactic category and the ontological category, respectively, with the same categorial index $\iota$, and $\delta$ is the operation of denotation.

From the principle $(CC)$ it follows that for any $e = f(e_1, e_2, \ldots, e_n) \in S$ with the main functor-function $f \in CATa/a_1a_2 \ldots a_n$ satisfying the condition $(CAT_f)$ the following conditions are satisfied:

$$\delta(f) \in ONTa/a_1a_2 \ldots a_n = ONTa^{ONTa_1 \times ONTa_2 \times \ldots \times ONTa_n} \tag{$ONTf$}$$

and

$$\delta(f(e_1, e_2, \ldots, e_n)) = \delta(f)(\delta(e_1), \delta(e_2), \ldots, \delta(e_n)). \tag{$PCD$}$$

The condition $(ONTf)$ states that the denotation (object reference) of the main functor of the composed *wfe* $e$ of the set $S$ is the set-theoretical function mapping the Cartesian product of ontological categories $ONTa_1 \times ONTa_2 \times \ldots \times ONTa_n$ into the ontological category $ONTa$ and it is defined by means of the condition $(PCD)$ connected with some Frege's ideas and called the principle of compositionality of denotation.

**3.3.1. An algebraic ontological structure corresponding to the partial algebra L.** The operation $\delta$ assigns the following ontological structure $\mathbf{R_L}$ of a reality corresponding to language $L$ to the algebraic structure $\mathbf{L}$:

$$\mathbf{R_L} = \langle ONT, ONT_{Fo} \rangle ,$$

where $ONT_{Fo}$ is the sum of all ontological categories corresponding to all functors of the set $Fo$. The structure $\mathbf{R_L}$ is a partial algebra similar to the algebra $\mathbf{L}$ and the principle ($PCD$) is simultaneously the condition of homomorphism of the algebra $\mathbf{L}$ into the algebra $\mathbf{R_L}$, i.e.

$$\delta : \langle S, Fo \rangle \xrightarrow[hom]{} \langle ONT, ONT_{Fo} \rangle \,.$$

A **model of language** $L$ is the structure of homomorphic images of components of $\mathbf{L}$, i.e. the substructure $\mathbf{M_L} = \langle \delta(S), \delta(Fo) \rangle$ of the structure $\mathbf{R_L}$.

If we distinguish in the set $B$ of basic *wfes* of $S$ the category $CATs$ of all sentences of language $L$, then the notion of *truthfulness* of any sentence $e \in CATs$ in the model $\mathbf{M_L}$ is defined as follows:

$$e \text{ is a } \textit{true sentence} \text{ in the model } \mathbf{M_L} \quad \text{iff} \quad \delta(e) \in T, \qquad (T)$$

where $T$ is primitive notion of the considered theory intuitively understood either as the singleton with the true value (in Freegan semantics) or as the set of all states of affairs that take place (in situational semantics).

## 4. The solution of the problem of quantifiers of $1^{st}$-order

The unsatisfactory efforts to establish, in the sense of the principle ($CC$) of categorial compatibility, the category of quantifiers in formalized $1^{st}$-order languages can be solved by means of notions and statements of the above outlined theory of categorial languages.

Let $L_1$ be any $1^{st}$-order formalized language. Let us treat any standard quantifier of $L_1$ as a context-dependent functor of two arguments:

1. a quantifier variable (the variable accompanying this quantifier) and
2. its scope, i.e. a sentential function including as a free variable the same variable as the quantifier variable.

### 4.1. Different types of the $1^{st}$-order quantifiers and their syntactic categories

A standard, the $1^{st}$-order quantifier is a functor forming a new sentential function (in particular a sentence of $L_1$) in which there occur one free variable less than in the scope of this quantifier (the variable bound by the quantifier). As such a functor, a quantifier can be treated as a set-theoretical function relative to the number of free individual variables occurring in its scope. So, we should not speak of one existential $\exists$ or one universal quantifier $\forall$ but about different types of such quantifiers depending of the number of free variables in their scope. We will use numerical superscripts in order to point out these different types of quantifiers.

Let

- $Var$ be the set of all individual variables for $L_1$, with categorial index $n_1$;
- $S = S_0$ – the set of all its sentences, with the categorial index $s$;
- $S_k (k \geq 1)$ – the set of all sentential functions in which exactly $k$ free variables occur, with the index $s_k$.

For example, if $\alpha(x_1, x_2, x_3) \in S_3$, where $x_1, x_2, x_3 \in \mathit{Var}$, then the expressions:

$$\forall^3 x_2 \alpha(x_1, x_2, x_3) \in S_2,$$
$$\exists^2 x_3 \forall^3 x_2 \alpha(x_1, x_2, x_3) \in S_1,$$
$$\forall^1 x_1 \exists^2 x_3 \forall^3 x_2 \alpha(x_1, x_2, x_3) \in S_0$$

and quantifiers $\forall^3$, $\exists^2$, $\forall^1$ belong to different syntactic categories with indices $s_2/n_1 s_3$, $s_1/n_1 s_2$, $s/n_1 s_1$, respectively.

More generally, the quantifiers $\forall^k$ and $\exists^k$ $(k \geq 1)$ are treated as the functors-functions:

$$\forall^k, \exists^k : \mathit{Var} \times S_k \to S_{k-1} \qquad (S_0 = S).$$

Thus, in accordance to $(\mathit{CATf})$, for $k > 0$ we have

$$(\mathit{CAT}\forall^k, \exists^k) \qquad \forall^k, \exists^k \in \mathit{CAT}s_{k-1}/n_1 s_k \qquad (s_0 = s),$$

and the principle of syntactic connection $(\mathit{SC})$ for them is satisfied.

Their denotations and ontological categories should be defined in such a way as to satisfied the principle $(\mathit{CC})$ of categorial compatibility (their denotations should belong to the ontological category $\mathit{ONT}s_{k-1}/n_1 s_k$) and the principle $(\mathit{PCD})$ of compositionality of denotation.

Let the denotation operation for the language $L_1$ be the function $d$ in Fregean, standard semantics and the function $\underline{d}$ in the situational, non-standard semantics:

$$d, \underline{d} : S(L_1) \to \mathit{ONT}(L_1)$$

mapping the set $S(L_1)$ of all *wfes* of $L_1$ into the set $\mathit{ONT}(L_1)$ which is the sum of all ontological categories in the ontological structure $\mathbf{R}_{L_1}$.

We will give here two possible solutions of denotations of quantifiers of the 1st-order taking into account two different ways of understanding of the denotation of sentences and sentential functions presented below.

### 4.2. Denotations of 1st-order quantifiers and their ontological categories

**4.2.1. Fregean semantics.** We assume that if $U$ is the universe of individuals in an established model $\mathbf{M}_{L_1}$ of $L_1$, 1 is the value of truth, 0 – the value of falsity then

$$d(x) \in \{U\} = \mathit{ONT}n_1 \qquad \text{for any } x \in \mathit{CAT}n_1 = \mathit{Var};$$
$$d(p) \in \{0, 1\} = \mathit{ONT}s \qquad \text{for any } p \in \mathit{CAT}s = S;$$
$$d(\mathit{sf}) \in 2^{U^k} = \mathit{ONT}s_k \qquad \text{for any } \mathit{sf} \in \mathit{CAT}s_k = S_k (k \geq 1)$$

and for any $x_1, x_2, \ldots, x_k \in \mathit{Var}$ and for any $\mathit{sf} = \alpha(x_1, x_2, \ldots, x_k) \in S_k$

$$d(\alpha(x_1, x_2, \ldots, x_k)) =$$
$$\{(u_1, u_2, \ldots, u_k) \in U^k \mid d(\alpha^o(x_1/u_1, x_2/u_2, \ldots, x_k/u_k)) = 1\},$$

where $\alpha^o(x_1/u_1, x_2/u_2, \ldots, x_k/u_k)$ is a sentence which we get from sentential function *sf* by replacement of its all free variables $x_1, x_2, \ldots, x_k$ of $\mathit{Var}$ by suitable individual names of individuals $u_1, u_2, \ldots, u_k$ of the universe $U$, i.e. the denotation of *sf* is the set of all $k$-tuples from $U^k$ which satisfy this sentential function.

Denotation for the quantifier $\forall^k$ of the type $k(k \geq 1)$ is defined by induction as follows:

a) for $k = 1$ and any $\alpha(x) \in S_1$

$$d(\forall^1 x\alpha(x)) = d(\forall^1)(d(x)), d(\alpha(x))) = \begin{cases} 1, & d(x) = U = d(\alpha(x)) \\ 0, & d(x) = U \neq d(\alpha(x)); \end{cases}.$$

According to a) the quantifier sentence obtained from any sentential function $\alpha(x)$ by preceding it with the universal quantifier $\forall^1$ is a true sentence in the established model $\mathbf{M}_{L_1}$ of $L_1$ with the universe of individuals $U$ *iff* every object of the universe $U$ satisfies the $\alpha(x)$ which is the scope of $\forall^1$.

b) for $k = j + 1(j > 0)$ and any $\alpha(x_1, x_2, \ldots, x, \ldots, x_{j+1}) \in S_{j+1}$

$$\begin{aligned} &d(\forall^{j+1} x\alpha(x_1, x_2, \ldots, x, \ldots, x_{j+1})) = \\ &\quad = d(\forall^{j+1})(d(x), d(\alpha(x_1, x_2, \ldots, x, \ldots, x_{j+1}))) = \\ &\quad = \{(u_1, u_2, \ldots, u_{j+1}) \in U^j \mid d(\alpha^o(x_1/u_1, x_2/u_2, \ldots, x/u, \ldots, x_{j+1}/u_{j+1})) = 1 \\ &\qquad \text{for each } u \in U\}. \end{aligned}$$

According to b) the denotation of the sentential function $sf_{k-1} \in S_{k-1}$ obtained from the sentential function $\alpha(x_1, x_2, \ldots, x, \ldots, x_{j+1}) \in S_k(k > 1)$ by binding the variable $x$ by the universal quantifier $\forall^k(k = j + 1 > 1)$ is the set of all $j = (k-1)$-tuples $(u_1, u_2, \ldots, u_{k-1})$ of individuals of $U$ such that all sentences obtained by the substitution of all $j$ free variables in $sf_{k-1}$, respectively, by names of individuals of these tuples and names of any individuals of $U$ representing $x$ are true; in other words the denotation of $sf_{k-1}$ is the set of all such $(k-1)$-tuples $(u_1, u_2, \ldots, u_{k-1})$ of individuals of $U$ that for any individual $u$ of $U$ $k$-tuples $(u_1, u_2, \ldots, u, \ldots, u_{k-1})$ build from them satisfy the scope $\alpha(x_1, x_2, \ldots, x, \ldots, x_{j+1})$ of the quantifier $\forall^k$.

Thus for any $k \geq 1$

$$d(\forall^k) \in ONTs_{k-1}/n_1 s_k = ONTs_{k-1}^{ONTn_1 \times ONTs_k}.$$

Similarly for $d(\exists^k)$:

a) for $k = 1$ and any $\alpha(x) \in S_1$

$$d(\exists^1 x\alpha(x)) = d(\exists^1)(d(x)), d(\alpha(x))) = \begin{cases} 1, & d(x) \cap d(\alpha(x)) \neq \emptyset \\ 0, & d(x) \cap d(\alpha(x)) = \emptyset; \end{cases}.$$

b) for $k = j + 1(j > 0)$ and any $\alpha(x_1, x_2, \ldots, x, \ldots, x_{j+1}) \in S_{j+1}$

$$\begin{aligned} &d(\exists^{j+1} x\alpha(x_1, x_2, \ldots, x, \ldots, x_{j+1})) = \\ &\quad = d(\exists^{j+1})(d(x), d(\alpha(x_1, x_2, \ldots, x, \ldots, x_{j+1}))) = \\ &\quad = \{(u_1, u_2, \ldots, u_{j+1}) \in U^j \mid d(\alpha^o(x_1/u_1, x_2/u_2, \ldots, x/u, \ldots, x_{j+1}/u_{j+1})) = 1 \\ &\qquad \text{for some } u \in U\}. \end{aligned}$$

According to a) the quantifier sentence obtained from any sentential function $\alpha(x)$ by preceding it with the existential quantifier $\exists^1$ is true sentence in the established model $\mathbf{M}_{L_1}$ of $L_1$ with the universe of individuals $U$ *iff* at least one object of the universe $U$ satisfies the $\alpha(x)$ which is the scope of $\exists^1$.

According to b) the denotation of the sentential function $sf_{k-1} \in S_{k-1}$ obtained from the sentential function $\alpha(x_1, x_2, \ldots, x, \ldots, x_{j+1}) \in S_k (k > 1)$ by binding the variable $x$ by the existential quantifier $\exists^k (k = j + 1 > 1)$ is the set of all $j = (k-1)$-tuples $(u_1, u_2, \ldots, u_{k-1})$ of individuals of $U$ such that all sentences obtained by the substitution of all $j$ free variables in $sf_{k-1}$, respectively, by names of individuals of these tuples and the substitution some individual name of $u$ for $x$ are true; in other words the denotation of $sf_{k-1}$ is the set of all such $(k-1)$-tuples $(u_1, u_2, \ldots, u_{k-1})$ of individuals of $U$ that for some individual $u$ of $U$ $k$-tuples $(u_1, u_2, \ldots, u, \ldots, u_{k-1})$ build from them satisfy the scope $\alpha(x_1, x_2, \ldots, x, \ldots, x_{j+1})$ of the quantifier $\exists^k$.

Thus, for any $k \geq 1$

$$d(\exists^k) \in ONTs_{k-1}/n_1 s_k = ONTs_{k-1}^{ONTn_1 \times ONTs_k}.$$

Moreover, the principle $(CC)$ is also valid for $\forall^k$ and $\exists^k$ in situational semantics.

**4.2.2. Situational semantics.** In situational semantic we assume that

$$\underline{d}(x) \in \{U\} = ONTn_1 \qquad \text{for any } x \in CATn_1 = Var;$$
$$\underline{d}(p) \in \{St\} = ONTs \qquad \text{for any } p \in CATs = S,$$

where $St$ is the set of all states of affairs, $St = T \cup F, T \cap F = \emptyset$ and $T$ is the nonempty set of all states of affairs that take place and $F$ – the nonempty set of remaining states of affairs. $St_k \subset St$ is the set of states of affairs with $k$ individuals.

$$\underline{d}(sf) \in 2^{St_k} = ONTs_k \qquad \text{for any } sf \in CATs_k = S_k$$

and for any $x_1, x_2, \ldots, x_k \in Var$ and for any $sf = \alpha(x_1, x_2, \ldots, x_k) \in S_k$

$$\underline{d}(\alpha(x_1, x_2, \ldots, x_k)) =$$
$$\{s \in St_k \mid s = \underline{d}(\alpha^0(x_1/u_1, x_2/u_2, \ldots, x_k/u_k)) \text{ for any } (u_1, u_2, \ldots, u_k) \in U^k\}.$$

So, if the denotation operation is understood here as the operation $\underline{d}$ then the denotations of sentences are states of affairs and the denotation of any sentential function is the set of all states of affairs that are denotations all sentences represented by the sentential function.

Denotation for the quantifier $\forall^k$ is defined by induction as follows:

a) for $k = 1$ and any $\alpha(x) \in S_1$

$$\underline{d}(\forall^1 x \alpha(x)) = \underline{d}(\forall^1)(\underline{d}(x)), \underline{d}(\alpha(x))) \in T \text{ iff } \underline{d}(\alpha^o(x/u)) \in T \text{ for each } u \in U;$$

b) for $k = j + 1 (j > 0)$ and any $\alpha(x_1, x_2, \ldots, x, \ldots, x_{j+1}) \in S_{j+1}$

$$\begin{aligned}&\underline{d}(\forall^{j+1} x\alpha(x_1, x_2, \ldots, x, \ldots, x_{j+1})) = \\ &\quad = \underline{d}(\forall^{j+1})(\underline{d}(x), \underline{d}(\alpha(x_1, x_2, \ldots, x, \ldots, x_{j+1}))) = \\ &\quad = \{s \in St \mid s = \underline{d}(\alpha^o(x_1/u_1, x_2/u_2, \ldots, x/u, \ldots, x_k/u_k)) \\ &\qquad \text{for each } u \in U, \text{ any } (u_1, u_2, \ldots, u_{j+1}) \in U^j\}.\end{aligned}$$

According to a) the quantifier sentence obtained from any sentential function $\alpha(x)$ by preceding it with the universal quantifier $\forall^1$ is a true sentence in an established model $\mathbf{M}_{L_1}$ of the language $L_1$ with the universe of individuals $U$ *iff* every sentence representing this sentential function is true (because their denotations are states of affairs that take place).

According to b) the denotation of sentential function $sf_{k-1} \in S_{k-1}$ obtained from the sentential function $\alpha(x_1, x_2, \ldots, x, \ldots, x_{j+1}) \in S_k (k > 1)$ by binding the variable $x$ by the universal quantifier $\forall^k$ is the set of all denotations of sentences (intuitively – the set of all states of affairs describing by these sentences) which can be obtained from $sf_{k-1}$ by replacing all free variables in it with individual names of any individuals of $U$; in other words, it is the set of all denotations of sentences (all states of affairs) which can be obtained from $\alpha(x_1, x_2, \ldots, x, \ldots, x_{j+1})$ by replacement for the variable $x$ binding by $\forall^k$ individual names of any individual of $U$ (of the denotation of this variable) and for remaining variables in it also individual names of any individuals of $U$.

Thus, for any $k \geq 1$

$$\underline{d}(\forall^k) \in ONTs_{k-1}/n_1 s_k = ONTs_{k-1}^{ONTn_1 \times ONTs_k}.$$

Similarly for $\underline{d}(\exists^k)$:

a) for $k = 1$ and any $\alpha(x) \in S_1$

$$\underline{d}(\exists^1 x\alpha(x)) = \underline{d}(\exists^1)(\underline{d}(x)), \underline{d}(\alpha(x))) \in T \quad \text{iff} \quad T \cap \underline{d}(\alpha(x)) \neq \emptyset$$

b) for $k = j + 1 (j > 0)$ and any $\alpha(x_1, x_2, \ldots, x, \ldots, x_{j+1}) \in S_{j+1}$

$$\begin{aligned}&\underline{d}(\exists^{j+1} x\alpha(x_1, x_2, \ldots, x, \ldots, x_{j+1})) = \\ &\quad = \underline{d}(\exists^{j+1})(\underline{d}(x), \underline{d}(\alpha(x_1, x_2, \ldots, x, \ldots, x_{j+1}))) = \\ &= \quad \{s \in St \mid s = \underline{d}(\alpha^o(x_1/u_1, x_2/u_2, \ldots, x/u, \ldots, x_{j+1}/u_{j+1})) \\ &\qquad \text{for some } u \in U, \text{ any } (u_1, u_2, \ldots, u_{j+1}) \in U^j\}.\end{aligned}$$

Thus, for any $k \geq 1$

$$\underline{d}(\exists^k) \in ONTs_{k-1}/n_1 s_k = ONTs_{k-1}^{ONTn_1 \times ONTs_k}.$$

**4.3. The syntactic and semantic compatibility of quantifiers**

In our categorial approach to syntax and semantics of the $1^{\text{st}}$-order formalized language $L_1$ its quantifiers have been treated as context-dependent two-argument functors-functions of different categorial types $k > 0$ (defined on the set *Var* of all its individual variables and the set of all its sentential functions $S_k$ with exactly

$k$ free variables) and with values in the set of sentential functions $S_{k-1}$ possessing one free variable less or, in particular, in the set of sentences $S$:

$$\forall^k, \exists^k : Var \times S_k \to S_{k-1} \qquad (S_0 = S).$$

Thus, according to the condition $(CATf)$, quantifiers $\forall^k$, $\exists^k$ belong to syntactic categories:

$$(CAT\forall^k, \exists^k) \qquad \forall^k, \exists^k \in CATs_{k-1}/n_1 s_k = CATs_{k-1}^{CATn_1 \times CATs_k} \qquad (s_0 = s),$$

and it means that they satisfy the principle $(SC)$ of syntactic connection.

It was also shown that for the denotation operations:

$$d, \underline{d} : S(L_1) \to ONT(L_1)$$

their denotations, according to the condition $(ONTf)$, belong to ontological categories:

$$d(\forall^k), \underline{d}(\forall^k), d(\exists^k), \underline{d}(\exists^k) \in ONTs_{k-1}/n_1 s_k = ONTs_{k-1}^{ONTn_1 \times ONTs_k}. \quad (ONT\forall^k, \exists^k)$$

## 5. Conclusions

From the conditions $(CAT\forall^k, \exists^k)$ and $(ONT\forall^k, \exists^k)$ follow the following conclusions:

1. *the 1*$^{st}$*-order quantifiers* $\forall^k, \exists^k (k > 0)$ *satisfy the principle of syntactic connection* $(SC)$ *and the principle of categorial compatibility* $(CC)$

   and
2. *the problem of standard quantifiers is solved by employing the conceptual apparatus and statements of the outlined theory of categorial languages.*

It should also be noted that

3. *in languages with other operators biding variables the problem of their denotations can be solved in an analogous way,*

   but
4. *for branching quantifiers used in Independence-Friendly logic (see Hintikka* [14]*) the outlined here denotational (compositional) semantics does not work.*

However,

5. *according to Frege's ideas, the proposed categorial approach to language syntax and semantics can be developed in the same spirit for formalized languages of higher order than 1.*
6. *the proposed approach to semantics of the 1st-order formalized languages of differ from the standard in the Tarski's approach* [29] *and other improved versions; first of all it refers to the concept of denotation of any language expression instead to the concept of satisfaction – the crucial ancillary notion in the definition of truth; this notion may be omitted in the definition of the concept of a true sentence and probably replaced by the notion of denotation.*

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