A Mystery of Grzegorczyk's Logic of Descriptions

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Abstract. In 2011, Andrzej Grzegorczyk formulated *Logic of Descriptions* (LD), a new logical system in which the classical equivalence has been replaced with the descriptive equivalence. Two sentences are descriptively equivalent whenever they describe the same state of affairs. Grzegorczyk's logic LD is built from the ground up by revising the axioms of classical propositional logic and rejecting those that do not correspond to the intended interpretation of the descriptive equivalence as the connective expressing equimeaning relations between sentences. Grzegorczyk's last paper, which introduced in detail philosophical motivations of LD and its axiomatization, has become an inspiration for investigating the properties of LD and its various modifications. In this paper we present the basics of Grzegorczyk's logic LD and then we survey the recent results on LD that have shed light on mysterious properties of the Grzegorczyk's descriptive equivalence connective.

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Motto:

It is important to approach the deeds of previous generations with the right attitude, accepting the fact that their actions were an actual step of their development. Regardless of any alternative ways in which reality could have evolved, we must not belittle their thoughts and actions, but rather augment them, keeping in mind all that our ancestors could not have taken into account. So, let us not claim that we are building a new philosophy on new principles, as we are not essentially wiser than our predecessors, but let us utilize the whole record of the past, carefully considering their whole evolution (biological and cultural). Their experiences form the basis of our growth.

Andrzej Grzegorczyk

1. Introduction

Andrzej Grzegorczyk devoted the last five years of his life to the study of a new logical system avoiding weaknesses and paradoxes of the classical logical connectives. His main goal was to build a logic determined by the logical structure and properties of descriptions of the world that people use in everyday life. Grzegorczyk emphasized very strongly the extremely positive role played by formal and metamathematical studies in the development of science in the twentieth century. He appreciated the particularly powerful impact of classical logic on the development of information technology and artificial intelligence. Yet at the same time, he felt that the time had come to reject the current paradigm and to propose another way of describing the meaning of logical concepts, not yet used by researchers, and based on the analysis of the role of human language in its entirety to the challenge of the human condition.

The main philosophical assumption of Grzegorczyk's standpoint was that in the human description of the cognized world's phenomena, the roles of negation, conjunction, and disjunction differ significantly from those of implication and equivalence. Negation, conjunction, and disjunction are very primitive and have clear intuitive descriptive meaning, while the classical implication and equivalence are derivative and have no intuitively plausible sense. Furthermore, it is exactly implication and equivalence that are responsible for some paradoxical laws of classical logic, such as "false implies everything", "truth is implied by anything" and "all true sentences are logically equivalent to each other".

As a consequence, states Grzegorczyk, we are forced to accept that among all the logical connectives exactly negation, conjunction, and disjunction—together with the *equimeaning connective* (or *descriptive equivalence*) expressing the assertion that two descriptions have the same meaning—are well suited as the primitive concepts of a new logic. Indeed, those four connectives are crucial and necessary for descriptive practice, that is, for the way people actually describe reality. As *descriptions* and *descriptive equivalences* among them have became crucial for Grzegorczyk's approach, he called his new logical system the *Logic of Descriptions,* or LD for short.

The first exposition of Grzegorczyk's new logic, its philosophical motivations and assumptions was published in 2011 in the paper [5], cf. [6]. In the paper, Grzegorczyk proposed a number of axioms and rules that the equimeaning connective (descriptive equivalence), denoted by $\equiv$, should satisfy and he posed a number of open problems, in particular whether the new connective $\equiv$ is different than the classical equivalence. Grzegorczyk's study on the logic of descriptions was quickly joined by other researchers, in particular the authors of this paper. In 2012 in the paper [3], we published the first results on the formal properties of LD, showing that the descriptive equivalence connective is essentially different than the classical one and the logic itself is indeed new. Further results are presented in the forthcoming paper [4].

Andrzej Grzegorczyk was informed about our results, he actively participated in seminars and meetings devoted to the logic of descriptions, where we had lively discussions about Grzegorczyk's motivations and the adequacy of his axioms. Toward the end of his life, Andrzej Grzegorczyk was working on a deeper exposition of the whole philosophical system underlying the logic LD. He was writing a book concerned with a complete philosophical theory of acquisition of logical connectives, making use of results from empirical sciences and linguistics. Unfortunately, professor Grzegorczyk did not finish this work before his death in 2014.

The aim of the present paper is to survey the current research results on Grzegorczyk's Logic of Descriptions LD and some of its variants. The paper consists of three sections and conclusions. In Section 2, we present the basics of the logic LD, its language, axiomatization and semantics. Section 3 is concerned with the exposition of the formal properties of LD, in particular those that we consider to be the most amazing and striking. In Section 4 we discuss some extensions and modifications of the logic LD. The paper ends with the conclusions section, in which we list a few open problems, among others.

2. The Logic of Descriptions LD

The logic LD is a propositional logic. The vocabulary of LD consists of the following pairwise disjoint sets of symbols:

- $\mathbb{V} = \{p_0, p_1, p_2, \ldots\}$ – an infinite countable set of propositional variables,
- $\{\neg, \wedge, \vee, \equiv\}$ – propositional operations of negation $\neg$, conjunction $\wedge$, disjunction $\vee$, and descriptive equivalence $\equiv$.

The set of LD is defined in a standard way as the smallest set that contains all the propositional variables and is closed under the propositional operations of LD. We will use the following four shorthand notations for LD-formulas:

- $(p \to q) \stackrel{\text{df}}{=} (\neg p \vee q)$ (classical implication)
- $(p \leftrightarrow q) \stackrel{\text{df}}{=} (p \to q) \wedge (q \to p)$ (classical equivalence)
- $(p \Rightarrow q) \stackrel{\text{df}}{=} (p \equiv (p \wedge q))$ (descriptive implication)
- $(p \Leftrightarrow q) \stackrel{\text{df}}{=} [(p \Rightarrow q) \wedge (q \Rightarrow p)]$ (quasi descriptive equivalence)

The logic LD has been defined by a Hilbert-style axiomatization given in [5]. It consists of 17 axioms and 4 rules of inference. The axioms are as follows:

(Ax0) $\neg(p \wedge \neg p)$
(Ax1) $p \equiv p$
(Ax2) $\neg\neg p \equiv p$
(Ax3) $p \equiv (p \wedge p)$
(Ax4) $p \equiv (p \vee p)$
(Ax5) $(p \wedge q) \equiv (q \wedge p)$
(Ax6) $(p \vee q) \equiv (q \vee p)$
(Ax7) $(p \wedge (q \wedge r)) \equiv ((p \wedge q) \wedge r)$

(Ax8) $(p \vee (q \vee r)) \equiv ((p \vee q) \vee r)$
(Ax9) $(p \wedge (q \vee r)) \equiv ((p \wedge q) \vee (p \wedge r))$
(Ax10) $(p \vee (q \wedge r)) \equiv ((p \vee q) \wedge (p \vee r))$
(Ax11) $\neg(p \wedge q) \equiv (\neg p \vee \neg q)$
(Ax12) $\neg(p \vee q) \equiv (\neg p \wedge \neg q)$
(Ax13) $(p \equiv q) \equiv (q \equiv p)$
(Ax14) $(p \equiv q) \equiv (\neg p \equiv \neg q)$
(Ax15) $(p \equiv q) \Rightarrow ((p \equiv r) \equiv (q \equiv r))$
(Ax16) $(p \equiv q) \Rightarrow ((p \wedge r) \equiv (q \wedge r))$
(Ax17) $(p \equiv q) \Rightarrow ((p \vee r) \equiv (q \vee r))$

The axioms express basic properties of conjunction, disjunction and negation with respect to the descriptive equivalence. Note that only one axiom does not involve the connective $\equiv$, namely the axiom (Ax0), which reflects a kind of consistency of the logic. The intended meanings of the other axioms are:

- Descriptive equivalence $\equiv$ represents an equivalence relation: (Ax1), (Ax13), and (Ax15).
- Extensionality property of $\equiv$: (Ax14) through (Ax17).
- Idempotency of conjunction and disjunction: (Ax3) and (Ax4).
- Commutativity of conjunction and disjunction: (Ax5) and (Ax6).
- Associativity of conjunction and disjunction: (Ax7) and (Ax8).
- Distributivity of conjunction over disjunction: (Ax9).
- Distributivity of disjunction over conjunction: (Ax10).
- Involution of negation: (Ax2).
- De Morgan laws for negation: (Ax11) and (Ax12).

The rules have the following forms:

$$(\mathsf{MPE})\quad \frac{\varphi \equiv \psi, \varphi}{\psi} \qquad (\mathsf{Sub})\quad \frac{\varphi(p_0, \dots, p_n)}{\varphi(p_0/\psi_0, \dots, p_n/\psi_n)}$$

$$(\wedge_1)\quad \frac{\varphi, \psi}{\varphi \wedge \psi} \qquad (\wedge_2)\quad \frac{\varphi \wedge \psi}{\varphi, \psi}$$

where $\varphi, \psi, \psi_0, \dots, \psi_n$ are any LD-formulas and $p_0, \dots, p_n$ are propositional variables with the additional restriction that the rule (Sub) applies only to axioms. The rules (Sub), $(\wedge_1)$, $(\wedge_2)$ are standard in classical logic. However, it should be emphasized that instead of the classical Modus Ponens rule, the logic LD has a similar inference rule but with respect to the descriptive equivalence operator. Note also that LD does not include any rule for introduction or elimination of disjunction and negation.

The provability of a formula is defined in LD in a standard way. Thus, a formula φ is said to be *provable in* LD ($\vdash \varphi$ for short) whenever there exists a finite sequence $\varphi_1, \dots, \varphi_n$ of LD-formulas, $n \geq 1$, such that $\varphi_n = \varphi$ and each φ_i, $i \in \{1, \dots, n\}$, is an axiom or follows from earlier formulas in the sequence by one of the rules of inference. If X is any set of LD-formulas, then φ is said to be LD-*provable from* X ($X \vdash \varphi$ for short) whenever there exists a finite sequence

$\varphi_1, \ldots, \varphi_n$ of LD-formulas, $n \geq 1$, such that $\varphi_n = \varphi$ and for each $i \in \{1, \ldots, n\}$, φ_i is an axiom or $\varphi_i \in X$ or φ_i follows from earlier formulas in the sequence by one of the rules of inference.

Observe that if we interpret $\equiv$ as the classical equivalence, then the LD-axioms are classical tautologies and all the rules preserve classical validity. It means that one of the possible models of LD is the two-element Boolean algebra of classical propositional logic. Therefore, $p \equiv \neg p$ is not provable in LD, and hence the logic LD is consistent in the sense that it does not entail all formulas. Moreover, as shown in [4], the logic LD is actually paraconsistent, that is, even a contradiction in LD does not entail all formulas. The proof of this fact uses semantics for LD.

The first sound and complete semantics for LD was introduced in [3]. Then, after some weakening and improvement of the LD-models, the strong soundness and completeness of LD with respect to the class of paraconsistent LD-models was proved in the paper [4]. Generally, models of the logic LD are based on the so-called *Grzegorczyk algebras* satisfying some further conditions. Now, following presentations given in [3] and [4], we present the definition of LD-models in detail.

A structure $(U, \oplus, \otimes)$ is said to be a *distributive bisemilattice* whenever the following hold, for all $a, b, c \in U$ and for any $\odot \in \{\otimes, \oplus\}$:

- $a \odot b = b \odot a$, (commutativity of $\otimes, \oplus$)
- $a \odot (b \odot c) = (a \odot b) \odot c$, (associativity of $\otimes, \oplus$)
- $a \odot a = a$, (idempotency of $\otimes, \oplus$)
- $a \oplus (b \otimes c) = (a \oplus b) \otimes (a \oplus c)$, (distributivity of $\otimes$ over $\oplus$)
- $a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c)$. (distributivity of $\oplus$ over $\otimes$)

A *de Morgan bisemilattice* is a structure $(U, \sim, \oplus, \otimes)$ such that $(U, \oplus, \otimes)$ is a distributive bisemilattice and for all $a, b \in U$, the following hold:

- $\sim\sim a = a$,
- $\sim(a \oplus b) = \sim a \otimes \sim b$.

A *Grzegorczyk algebra* is a structure $(U, \sim, \oplus, \otimes, \circ)$ such that $(U, \sim, \oplus, \otimes)$ is a de Morgan bisemilattice and for all $a, b, c \in U$, the following hold:

- $a \circ b = b \circ a$,
- $a \circ b = \sim a \circ \sim b$,
- $a \circ b = (a \circ b) \otimes ((a \circ c) \circ (b \circ c))$,
- $a \circ b = (a \circ b) \otimes ((a \oplus c) \circ (b \oplus c))$,
- $a \circ b = (a \circ b) \otimes ((a \otimes c) \circ (b \otimes c))$.

Fact 2.1.
A structure $(U, \sim, \oplus, \otimes, \circ)$ is a Grzegorczyk algebra if and only if the following conditions hold, for all $a, b, c \in U$:

(LD1) $a \circ b = b \circ a$,
(LD2) $a \circ b = (a \circ b) \otimes ((a \circ c) \circ (b \circ c))$,
(LD3) $a \circ b = \sim a \circ \sim b$,
(LD4) $a \circ b = (a \circ b) \otimes ((a \oplus c) \circ (b \oplus c))$,
(LD5) $a \circ b = (a \circ b) \otimes ((a \otimes c) \circ (b \otimes c))$,
(LD6) $a \oplus b = b \oplus a$,
(LD7) $a \oplus (b \oplus c) = (a \oplus b) \oplus c$,
(LD8) $a \oplus a = a$,
(LD9) $a \otimes b = b \otimes a$,
(LD10) $a \otimes (b \otimes c) = (a \otimes b) \otimes c$,
(LD11) $a \otimes a = a$,
(LD12) $a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c)$,
(LD13) $a \oplus (b \otimes c) = (a \oplus b) \otimes (a \oplus c)$,
(LD14) $\sim(a \oplus b) = \sim a \otimes \sim b$,
(LD15) $\sim(a \otimes b) = \sim a \oplus \sim b$,
(LD16) $\sim\sim a = a$.

It is worth to emphasize the following fact:

Fact 2.2. *Boolean algebras, Kleene algebras, and de Morgan algebras are Grzegorczyk algebras.*

The converse of the above does not hold. The class of Grzegorczyk algebras contains subclasses that form bases for semantics of various non-classical logics of different types. This shows how different from other well known non-classical logics the logic LD is. Grzegorczyk algebras are a base for structures of LD.

Definition 2.3. A *paraconsistent* LD*-structure* is a structure of the form

$$\langle U, \sim, \oplus, \otimes, \circ, D\rangle, \text{where}$$

- U, D are non-empty sets such that $D \subsetneq U$,
- $\langle U, \sim, \oplus, \otimes, \circ\rangle$ is a Grzegorczyk algebra,
- for all $a, b \in U$, the following hold:
  - $a \otimes b \in D$ if and only if $a \in D$ and $b \in D$,
  - $a \circ b \in D$ if and only if $a = b$,
  - $\sim(a \otimes \sim a) \in D$.

A paraconsistent LD-structure is called a *classical* LD*-structure* if there is no $a \in U$ such that $a \otimes \sim a \in D$. In what follows, paraconsistent LD-structures will be also referred to as LD-structures or LD-models.

Definition 2.4. Let $\mathcal{M} = (U, \sim, \oplus, \otimes, \circ, D)$ be an LD-structure. A *valuation* on $\mathcal{M}$ is any mapping $v\colon \mathbb{V} \to U$ such that for all LD-formulas φ and ψ:

- $v(\neg\varphi) = \sim v(\varphi)$,
- $v(\varphi \wedge \psi) = v(\varphi) \otimes v(\psi)$,
- $v(\varphi \vee \psi) = v(\varphi) \oplus v(\psi)$,
- $v(\varphi \equiv \psi) = v(\varphi) \circ v(\psi)$.

A formula φ is said to be *satisfied in* $\mathcal{M}$ *by a valuation* v if and only if $v(\varphi) \in D$. It is *true in* $\mathcal{M}$ whenever it is satisfied in $\mathcal{M}$ by all the valuations on $\mathcal{M}$, and it is LD-*valid* if it is true in all paraconsistent LD-structures.

Definition 2.5. Let X and φ be a set of LD-formulas and a single LD-formula, respectively. The formula φ is a *semantic consequence* of X, denoted by $X \models_{\mathsf{LD}} \varphi$, if for every paraconsistent LD-structure $\mathcal{M}$ and every valuation v in $\mathcal{M}$ such that $\mathcal{M}, v \models X$, it holds that $\mathcal{M}, v \models \varphi$.

In [3] soundness and completeness with respect to the classical LD-structures is proved, while in [4] it has been proved that LD is strongly sound and complete with respect to the paraconsistent LD-structures. Hence, the following holds:

Theorem 2.6 (Soundness and Completeness of LD). *For every* LD*-formula* φ *the following conditions are equivalent:*

1. φ *is* LD*-provable.*
2. φ *is* LD*-valid.*
3. φ *is true in all classical* LD*-structures.*

Theorem 2.7. *Let* X *and* φ *be a set of* LD*-formulas and a single* LD*-formula, respectively. Then, the following conditions are equivalent:*

1. $X \vdash_{\mathsf{LD}} \varphi$.
2. $X \models_{\mathsf{LD}} \varphi$.

The above theorem provides a way of proving that the logic LD is paraconsistent in the following sense:

Definition 2.8. A logic is *paraconsistent* (or *contradiction-tolerant*) iff there are formulas φ, ψ such that $\varphi \wedge \neg\varphi \nvdash \psi$.

In [4] the following theorem is proved:

Theorem 2.9. *The logic* LD *is paraconsistent. In particular,* $p \wedge \neg p \nvdash_{\mathsf{LD}} q$.

Proof. Let $\mathcal{M} = (U, \sim, \oplus, \otimes, \circ, D)$ be a paraconsistent LD-structure defined as:

$$\begin{aligned} U &= \{0,1,2\}, \\ \sim a &= 2 - a, \\ a \oplus b &= \max(a,b), \\ a \otimes b &= \min(a,b), \\ a \circ b &= \begin{cases} 1, & \text{if } a = b, \\ 0, & \text{otherwise;} \end{cases} \\ D &= \{1,2\}. \end{aligned}$$

Now, let $v(p) = 1$, $v(q) = 0$. Then, it is easy to show that $v(p \wedge \neg p) = 1 \in D$, but $v(q) = 0 \notin D$. Hence, the logic LD is paraconsistent. $\square$

The result on the semantics shows that LD has much in common with the non-Fregean logic SCI introduced by Suszko in [8], cf. [9]. Indeed, the logic LD can be seen as *non-Fregean* in the sense that it rejects the main assumption of classical Fregean logic, according to which sentences with the same truth value have the same denotations. Although the logics LD and SCI share the language and have similar philosophical motivations, they differ considerably in the formalization and they are *different logics,* as we will see.

3. Properties of LD

The first results on provable and unprovable formulas presented in [3] have shown in particular that: (1) the logic LD is new, in the sense that its class of valid formulas cannot be identified with valid formulas of any other well known non-classical logic, (2) the descriptive equivalence and implication connectives are different from the classical ones, (3) neither the absorption nor the boundedness laws are provable in LD.

In [3] the following has been proved:

Theorem 3.1. *The following formulas are not provable in* LD*:*

1. $(\varphi \equiv \psi) \leftrightarrow (\varphi \leftrightarrow \psi)$
2. $(\varphi \Rightarrow \psi) \leftrightarrow (\varphi \rightarrow \psi)$
3. $(\varphi \equiv \varphi) \equiv (\psi \equiv \psi)$
4. $(\varphi \vee (\varphi \wedge \psi)) \equiv \varphi$
5. $(\varphi \wedge (\varphi \vee \psi)) \equiv \varphi$
6. $(\varphi \vee \neg\varphi) \equiv (\psi \vee \neg\psi)$
7. $(\varphi \wedge \neg\varphi) \Rightarrow \psi$
8. $\varphi \Rightarrow (\psi \vee \neg\psi)$
9. $\neg(\varphi \equiv \neg\varphi)$

Proof. By way of example, we will show that the absorption laws, i.e., formulas 4 and 5 of the theorem, are not provable in LD.

Let $\mathcal{M} = (U, \sim, \oplus, \otimes, \circ, D)$ be a paraconsistent LD-structure such that $U = \{0, 1, 2, 3\}$, $D = \{2, 3\}$ and the operations are defined as follows:

$\sim$	0	1	2	3
	3	2	1	0

$\circ$	0	1	2	3
0	3	0	0	0
1	0	3	0	0
2	0	0	3	0
3	0	0	0	3

$\otimes$	0	1	2	3
0	0	0	0	0
1	0	1	1	0
2	0	1	2	3
3	0	0	3	3

$\oplus$	0	1	2	3
0	0	0	3	3
1	0	1	2	3
2	3	2	2	3
3	3	3	3	3

It can be verified that the above tables indeed define a Grzegorczyk algebra. Then, the absorption laws do not hold, as $1 \oplus (1 \otimes 0) = 1 \otimes (1 \oplus 0) = 0 \neq 1$. Therefore, the formulas $(p \vee (p \wedge q)) \equiv p$ and $(p \wedge (p \vee q)) \equiv p$ are not satisfied in $\mathcal{M}$ by a valuation v such that $v(p) = 1$ and $v(q) = 0$, which by Theorem 2.6 means that these formulas are not provable in LD. Proofs for other formulas can be found in [3]. □

Observe the following particular consequences of Theorem 3.1:

- The descriptive equivalence and implication are different from the classical ones (unprovability of formulas 1 and 2).
- True descriptive equivalences may be not identical (unprovability of formula 3).
- The absorption laws are not theorems of LD (unprovability of formulas 4 and 5).
- True sentences are not necessarily descriptively equivalent (unprovability of formula 6).
- False does not imply everything (unprovability of formula 7).
- Not everything implies the truth (unprovability of formula 8).
- Negation and disjunction do not behave in a classical way, which is clearly manifested by the unprovability of formula 9. Indeed, $\neg(p \equiv \neg p)$ is not a theorem of LD, while it can be easily observed that the formula $p \equiv \neg p$ is not satisfiable in any LD-model and, on the other hand, the formula $\neg(p \equiv \neg p) \vee (p \equiv \neg p)$ is valid in LD.

This shows that LD can be seen as a logic that avoids the classical paradoxes of implication. On the other hand, LD shares many of the classical laws.

Theorem 3.2. *The following formulas are* LD*-valid:*

1. $\varphi \vee \neg\varphi$
2. $(\varphi \wedge (\varphi \rightarrow \psi)) \rightarrow \psi$[1]
3. $(\varphi \equiv \psi) \Rightarrow (\varphi \Leftrightarrow \psi)$

Moreover, $(\varphi \Leftrightarrow \psi) \vdash_{\mathsf{LD}} (\varphi \equiv \psi)$ *and the following rules are admissible in* LD*:*

$$(\mathsf{tran})\quad \frac{\varphi \equiv \psi, \psi \equiv \theta}{\varphi \equiv \theta} \qquad\qquad (\vee)\quad \frac{\varphi}{\varphi \vee \theta}$$

The proofs of the above properties can be found in [3] and [4].

The LD-validity of the formula $(\varphi \equiv \psi) \Rightarrow (\varphi \Leftrightarrow \psi)$ and the fact that $(\varphi \Leftrightarrow \psi) \vdash_{\mathsf{LD}} (\varphi \equiv \psi)$ connect the descriptive equivalence $\equiv$ with the quasi descriptive equivalence operator $\Leftrightarrow$. Recall that it is defined as follows:

$$(p \Leftrightarrow q) \overset{\mathrm{df}}{\equiv} [(p \Rightarrow q) \wedge (q \Rightarrow p)]$$

As in classical logic, it could be expected that the descriptive equivalence can be expressed as the conjunction of two implications, which would mean that the descriptive and quasi descriptive equivalences are indistinguishable in LD. However, as shown in [4] it does not hold, since the following theorem is true:

Theorem 3.3. *The following formula is not* LD*-valid:*

$$(p \equiv q) \equiv (p \Leftrightarrow q).$$

[1] Recall that $\varphi \rightarrow \psi$ is short for $\neg\varphi \vee \psi$.

In [4] the following specific formulas have been discussed:

$$[(p \equiv q) \wedge (q \equiv r)] \Rightarrow (p \equiv r) \tag{AxT}$$
$$((p \equiv q) \wedge p) \equiv ((p \equiv q) \wedge q) \tag{AxD}$$

The formula (AxT) expresses a strong form of transitivity of $\equiv$. The formula (AxD) is called the *Delusion Axiom* in reference to an example used to illustrate its intuitive content. Consider someone who believes that Vladimir Putin and Donald Trump are the the same person, just wearing different masks. Such a deluded person should accept that the following two statements say the same:

(α) *Putin is the president of Russia*
(β) *Trump is the president of Russia*

Consequently, she would accept that the following statements are descriptively equivalent:

1. $\alpha \equiv \beta$ and α.
2. $\alpha \equiv \beta$ and β.

The acceptance of the identity of these claims above means evaluating a conjunction involving a delusion of identity as if the identity were true, which is expressed by axiom (AxD).

In [4] is has been shown that neither (AxT) nor (AxD) is a theorem of LD.

Theorem 3.4. *The following formulas are not provable in* LD*:*

1. $[(p \equiv q) \wedge (q \equiv r)] \Rightarrow (p \equiv r)$
2. $((p \equiv q) \wedge p) \equiv ((p \equiv q) \wedge q)$

There has been some controversy over the last three axioms of LD:

(Ax15) $(p \equiv q) \Rightarrow ((p \equiv r) \equiv (q \equiv r))$
(Ax16) $(p \equiv q) \Rightarrow ((p \wedge r) \equiv (q \wedge r))$
(Ax17) $(p \equiv q) \Rightarrow ((p \vee r) \equiv (q \vee r))$

One of their intended meanings is to express the extensionality principle for descriptions:

> Sentences that have equal meaning (have the same content or are the same descriptions) are interchangeable in all possible contexts.

Axioms (Ax15), (Ax16), (Ax17) seem to express this property in a stronger form. Their adequacy has been deeply discussed in [2] and the conclusion presented there is rather negative. Instead a new very weak *Minimal Grzegorczyk Logic* MGL has been introduced with a rule expressing the extensionality principle instead of the axioms. However, this account goes beyond the scope of this paper, so the interested reader should consult [2].

On the other hand, in the paper [4] other ways of expressing the extensionality principle have been explored. We start this discussion with the formal definitions.

Definition 3.5. The *Weak Extensionality Principle* (WEP) is the following meta-rule:

$$\frac{\varphi \equiv \psi}{\vartheta(p/\varphi) \equiv \vartheta(p/\psi)},$$

where φ, ψ, and ϑ are arbitrary formulas.

Definition 3.6. The *Strong Extensionality Principle* (SEP) is the claim

$$\vdash (\varphi \equiv \psi) \Rightarrow (\vartheta(p/\varphi) \equiv \vartheta(p/\psi)),$$

where φ and ψ are arbitrary formulas, and $\vartheta(p)$ is a formula in which p actually occurs.

Definition 3.7. The *Grzegorczyk Extensionality Principle* (GEP) is the statement

$$\vdash (\bar{\varphi} \equiv \bar{\psi}) \Rightarrow (\vartheta(p_1/\varphi_1, \ldots, p_n/\varphi_n) \equiv \vartheta(p_1/\psi_1, \ldots, p_n/\psi_n)),$$

where $\varphi_1, \ldots, \varphi_n, \psi_1, \ldots, \psi_n$ are arbitrary formulas, ϑ is a formula whose propositional variables are contained in $\{p_1, \ldots, p_n\}$, and $\bar{\varphi} \equiv \bar{\psi}$ is short for $(\varphi_1 \equiv \psi_1) \wedge \ldots \wedge (\varphi_n \equiv \psi_n)$.

Observe that SEP implies the WEP and the GEP implies the WEP for a given logic. In [4] the following have been proved

Theorem 3.8.

1. *Neither the SEP nor the GEP holds for* LD.
2. *The WEP holds for* LD.

In the next section we will discuss extensions of LD with axioms (AxT) and (AxD) as well as some other modifications of LD, together with their properties and mutual relationships.

4. Extensions and modifications of LD

In the previous section we discussed the most important properties of LD. In particular, we mentioned that the transitivity axiom (AxT) and the delusion axiom (AxD) are not provable in LD. Hence, it was quite natural to consider extensions of LD with these axioms, which were deeply studied in the paper [4].

Logics obtained by adding (AxT) and (AxD) to the axiomatization of LD are denoted by LDT and LDD, respectively. The notions of provability in LDT and LDD are defined in a similar way as in LD. Models of LDT and LDD are defined as paraconsistent LD-models that satisfy the semantic counterparts of axioms (AxT) and (AxD), respectively. Then, the notions of satisfaction, truth and validity in LDT and LDD are defined in a standard way.

Theorem 4.1.

1. $\vdash_{\mathsf{LDD}}$ (AxT).
2. $\nvdash_{\mathsf{LDT}}$ (AxD).
3. $\vdash_{\mathsf{LDT}} (p \equiv q) \equiv (p \Leftrightarrow q)$.

The above theorem is proved in [4]. Note that the last formula is not provable in LD, while it is in LDT. Thus, LDT seems to be a better candidate than LD to compare Grzegorczyk's approach with various other non-classical logics, which are usually defined mainly in terms of implication.

In [4] the following is proved:

Theorem 4.2.

1. $\nvdash_{\mathsf{LD}} p \Rightarrow (p \vee q)$.
2. $\nvdash_{\mathsf{LDT}} p \Rightarrow (p \vee q)$.
3. $\nvdash_{\mathsf{LDD}} p \Rightarrow (p \vee q)$.

The above theorem allows us to show that LD, LDT, and LDD are different from several non-classical logics. Recall that $p \rightarrow (p \vee q)$ is provable in intuitionistic logic and relevance logics T, E, R, EM, and RM, see e.g.,[1]. Hence, if we identify the descriptive implication $\Rightarrow$ with the implication of the other logic, then LD, LDT, and LDD are different from any of the aforementioned non-classical logics. Moreover, the descriptive equivalence $\equiv$ cannot be identified with a necessary equivalence in any class of Kripke frames since $\Box(p \leftrightarrow (p \wedge (p \vee q)))$ is true in all frames.

Definition 4.3. Let L and L' be logics. A logic L is said to be:

- *not stronger than* L', $\mathsf{L} \leq \mathsf{L}'$ for short, whenever all formulas valid in L are valid in L',
- *equal with* L', $\mathsf{L} = \mathsf{L}'$ for short, $\mathsf{L} \leq \mathsf{L}'$ and $\mathsf{L}' \leq \mathsf{L}$,
- *weaker than* L', $\mathsf{L} < \mathsf{L}'$ for short, whenever $\mathsf{L} \leq \mathsf{L}'$ and $\mathsf{L} \neq \mathsf{L}'$,
- *uncomparable with* L' whenever $\mathsf{L} \not\leq \mathsf{L}'$ and $\mathsf{L}' \not\leq \mathsf{L}$.

Comparing provable formulas in the logics LD, LDT, and LDD, we can state the following:

Fact 4.4. $\mathsf{LD} < \mathsf{LDT} < \mathsf{LDD}$.

As noted in the previous section, the question whether the last three axioms of LD adequately express the extensionality principle has caused some controversy. This has led to some modifications of LD in which these three axioms have been substituted with alternative forms. The most important ones, explored in the paper [4], are the logics LE and LDS obtained by replacing the axioms (Ax15), (Ax16), (Ax17) with the following formulas, respectively:

$(\mathsf{Ax15})_{\mathsf{LE}}$ $((p \equiv q) \wedge (p \equiv r)) \equiv ((p \equiv q) \wedge (q \equiv r))$
$(\mathsf{Ax16})_{\mathsf{LE}}$ $((p \equiv q) \wedge (p \wedge r)) \equiv ((p \equiv q) \wedge (q \wedge r))$
$(\mathsf{Ax17})_{\mathsf{LE}}$ $((p \equiv q) \wedge (p \vee r)) \equiv ((p \equiv q) \wedge (q \vee r))$

$(\mathsf{Ax15})_{\mathsf{LDS}}$ $((p \equiv q) \wedge (r \equiv s)) \Rightarrow ((p \equiv r) \equiv (q \equiv s))$
$(\mathsf{Ax16})_{\mathsf{LDS}}$ $((p \equiv q) \wedge (r \equiv s)) \Rightarrow ((p \wedge r) \equiv (q \wedge s))$
$(\mathsf{Ax17})_{\mathsf{LDS}}$ $((p \equiv q) \wedge (r \equiv s)) \Rightarrow ((p \vee r) \equiv (q \vee s))$

The axioms for LDS are essentially copied from Suszko's Sentential Calculus of Identity, substituting the descriptive implication for the classical one.

Below we list some properties of the logics under consideration. Their proofs can be found in [4].

Theorem 4.5.

1. *The formula* $(p \equiv q) \equiv ((p \equiv q) \wedge (q \equiv q))$ *is provable in* LE*, but not in* LD*.*
2. LE $\leq$ LDD*.*
3. LE $\not\leq$ LDT*.*
4. LE *is uncomparable with* LD *and* LDT*.*
5. LDS *is paraconsistent.*
6. LDS *and* LD *are uncomparable.*
7. LDS $<$ LDT*.*
8. *The GEP holds for* LDS*.*
9. *The SEP holds for* LDT*.*
10. *The WEP does not hold for* LE*.*

All these results lead to the following:

Theorem 4.6.

1. LD $<$ LDT $<$ LDD
2. LDS $<$ LDT $<$ LDD
3. LE $<$ LDD

Hence, we have the following picture of dependencies among the logics LD, LDT, LDD, LE, LDS:

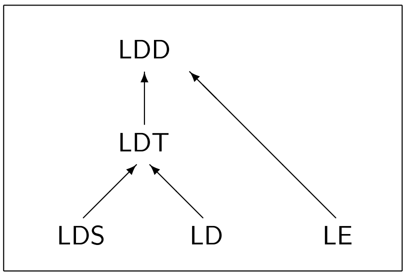

TABLE 1. The semilattice of Grzegorczyk's logics of descriptions.

We end this section with two summary tables:

	LD	LDT	LDD	LE	LDS
WEP	+	+	+	−	+
SEP	−	+	+	−	−
GEP	−	+	+	−	+

Table 2. Extensionality principles in Grzegorczyk's logics of descriptions.

	(AxT)	(AxD)	$(p \equiv q) \equiv (p \Leftrightarrow q)$	LD	LE	LDS
LD	−	−	−	+	−	−
LDT	+	−	+	+	−	+
LDD	+	+	+	+	+	+
LE	−	?	?	−	+	−
LDS	−	−	?	−	−	+

Table 3. Validity of some special axioms.

5. Conclusions

We have discussed the logic LD of descriptions together with its extensions LDT and LDD as well as the modifications LE and LDS. We have presented a strongly sound and complete semantics for LD, and used it to show that LD is a paraconsistent logic. Then, we have listed examples of classical laws which are provable and not provable in LD. We have reported what extensionality properties the logic LD has. Finally, we have presented formal properties of extended and modified versions of LD and stated the relationships among them.

Still there are many open problems concerning the logics in question. Below we list a few of them. In what follows, L stand for any of LD, LE, LDS, LDT, or LDD.

1. Is L decidable? If so, what is its complexity?
2. Is L equivalent to a previously known paraconsistent logic? If not, how does it relate to them?
3. How is a logic obtained by replacing $\equiv$ with $\Leftrightarrow$ in L related to L?
4. The last three axioms of LE express a property that resembles extensionality. Can this property and its relationship with actual extensionality be formulated in an informal, intuitive way?

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