# Post algebras in the work of Helena Rasiowa

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**Abstract.** A survey of some classes of Post algebras is given including the class of plain semi-Post algebras, Post algebras of order m, m>1, as its particular instance, Post algebras of order $\omega^+$ , and Post algebras of order $\omega + \omega^*$. Representation theorems for each of the classes are given. Some examples of the algebras in the classes are constructed.

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Emil Post doctoral dissertation [24] (see also Post [25]) provided a description of an $n$-valued, $n > 1$, functionally complete algebra. The notion of Post algebra was introduced in Rosenbloom [43]. Then was the paper by Wade [53] and in Rousseau [44, 45] an equivalent formulation of Post algebra was given which became a starting point for an extensive research. Various generalizations of Post algebras inspired by computer science have been proposed. In 1970-ies Helena Rasiowa and Tadeusz Traczyk run a seminar on Post algebras in the Department of Mathematics of the University of Warsaw, gathering the participants both from the Polish universities as well as guests and PhD students from abroad. This paper is a survey of major classes of Post algebras which were the subject of research of Helena Rasiowa at that time and were studied by the participants of the seminar. The present paper is based on the earlier paper included in:

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## 1. Plain semi-Post algebras

These algebras were introduced and investigated in Cat Ho [1], Cat Ho and Rasiowa [2–4]. Let $(T, \leqslant)$ be a poset. A subset $s$ of $T$ is an ideal provided that $s \neq \emptyset$ and for all $t \in s$ and $w \in T$, if $w \leqslant t$ then $w \in s$. Let $ET$ be the set of ideals of $T$ together with the empty set $\emptyset$. Clearly, $T \in ET$. It is known that any $s \in ET$ is of the form $s = \bigcup\{s(t) : s(t) \subseteq s\}$, where $s(t) = \{w \in ET : w \leqslant t\}$. The system $(ET, \subseteq)$ is a complete lattice, where join and meet are set-theoretical union and intersection, respectively.

An abstract algebra

(P) $\mathbf{P} = (P, \cup, \cap, \to, \neg, \{d_t : t \in T\}, \{e_s : s \in ET\})$

where $\cup, \cap, \to$ are 2-argument operations, $\neg$, $d_t$ for $t \in T$ are unary operations and $e_s$ for $s \in ET$ are 0-argument operations (constants) is a plain semi-Post algebra (psP-algebra) of type $T$ provided that the following conditions are satisfied:

(p0) $(P, \cup, \cap, \to, \neg)$ is a Heyting algebra with the zero element $\mathbf{0} = e_\emptyset$ and the unit element $\mathbf{1} = e_T$,

for any $a, b \in P$

(p1) $d_t(a \cup b) = d_t a \cup d_t b$,

(p2) $d_t(a \cap b) = d_t a \cap d_t b$,

(p3) $d_w d_t a = d_t a$,

(p4) $d_t e_s = \mathbf{1}$ if $t \in s$, otherwise $d_t e_s = \mathbf{0}$,

(p5) $d_t a \cup \neg d_t a = \mathbf{1}$,

(p6) $a = \bigcup\{e_{s(t)} \cap d_t a : t \in T\}$ where $\bigcup$ is the least upper bound in $P$.

Let $\mathbf{P} = (P, \cup, \cap, \to, \neg, \{d_t : t \in T\}, \{e_s : s \in ET\})$ be a psP-algebra of type $(T, \leqslant)$. By $B_P$ we denote the set of elements of $P$ of the form $d_t a$, $t \in T$.

**Proposition 1.1.**

(a) *$B_P$ is closed under the operations $\cup$, $\cap$, $\to$, $\neg$ of $P$.*

(b) *The algebra $\mathbf{B}_P = (B_P, \cup, \cap, \to, \neg, \mathbf{1}, \mathbf{0})$ is a Boolean algebra.*

Let $C_P$ be the set of all complemented elements in the distributive lattice $(P, \cup, \cap)$. Then

**Proposition 1.2.**

(a) *$C_P$ is closed under the operations $\cup$, $\cap$, $\to$, $\neg$ of $P$.*

(b) *The algebra $\mathbf{C}_p = (C_P, \cup, \cap, \to, \neg, \mathbf{1}, \mathbf{0})$ is a Boolean algebra.*

(c) *For every* $a \in C_P$, $d_t\neg a = \neg d_t a$, $t \in T$.

Note that $B_P$ and $C_P$ do not always equal. Consider a poset $(T, \leqslant)$ such that $T = \{a, b, c\}$ and $\leqslant = \{(b, a)\}$. Then $ET = \{\emptyset, \{b\}, \{c\}, \{b, c\}, \{a, b\}, T\}$, $B_P = \{\emptyset, T\}$, and $C_P = \{\emptyset, T, \{c\}, \{a, b\}\}$.

**Proposition 1.3 (Epstein lemma).** *For any set* $\{a_j : j \in J\}$ *of elements in* $P$ *it holds*

(a) $a = \bigcup^P \{a_j : j \in J\}$ *iff for every* $t \in T$ $d_t a = \bigcup^{B_P} \{d_t aj : j \in J\}$,

(b) $a = \bigcap^P \{a_j : j \in J\}$ *iff for every* $t \in T$ $d_t a = \bigcap^{B_P} \{d_t aj : j \in J\}$,

*where* $\bigcup^P$, $\bigcap^P$, $\bigcup^{B_P}$, $\bigcap^{B_P}$ *denote infinite joins and meets in the algebras* $\mathbf{P}$ *and* $\mathbf{B}_P$, *respectively.*

**Proposition 1.4.**

(a) $d_t(a \to c) = \bigcap\{d_w a \to d_w c : w \leqslant t\}$

(b) $d_t\neg a = \bigcap\{\neg d_w a : w \leqslant t\}$

(c) $d_w a \leqslant d_t a$ *whenever* $w \leqslant t$, *for any* $w, t \in T$

(d) $a \leqslant b$ *iff* $d_t a \leqslant d_t b$ *for all* $t \in T$

(e) $e_w \leqslant e_t$ *iff* $w \subseteq t$, *for any* $w, t \in ET$

It follows that every psP-algebra of type $(T, \leqslant)$ uniquely determines a set of infinite meets of $P$

$$M(P) = \left\{\bigcap\{d_w a \to d_w c : w \leqslant\} : t \in T\right\}.$$

Observe that for any sets $s', s'' \in ET$ there exists the relative pseudo-complement $s' \to s''$ defined by

$$s' \to s'' = \bigcup\{s \in ET : s' \cap a \subseteq s''\}$$

and the pseudo-complement $\neg s'$ defined by

$$\neg s' = s' \to \emptyset = \bigcup\{s \in ET : s' \cap s = \emptyset\}.$$

Clearly, $s' \to s'', \neg s' \in T$.

**Proposition 1.5.**

(a) *For any poset* $(T, \leqslant)$, *the system* $(ET, \cup, \cap, \to, \neg, T, \emptyset)$, *where* $\cup$, $\cap$ *are set-theoretical operations of union and intersection, respectively, and* $\to$, $\neg$ *are defined as above, is a Heyting algebra with the unit element* $T$ *and zero element* $\emptyset$.

(b) *Given a psP-algebra* $\mathbf{P} = (P, \cup, \cap, \to, \neg, \{d_t : t \in T\}, \{e_s : s \in ET\})$, *let* $EP = \{e_s : s \in ET\}$. *Then* $(EP, \leqslant)$ *is a poset isomorphic to* $(ET, \subseteq)$.

Condition (b) follows from Proposition 1.4*(e)*.

*Example* 1.1. An important example of a psP-algebra is the following algebra, referred to as a basic psP-algebra:

$$(ET, \cup, \cap, \rightarrow, \neg, \{d_t : t \in T\}, \{e_s : s \in ET\})$$

where $(ET, \cup, \cap, \rightarrow, \neg)$ is the Heyting algebra defined above and the operations $d_t, t \in T$, and $e_s, s \in ET$, are defined by:

$$e_s = s, \text{ in particular } e_\emptyset = \emptyset \text{ and } e_\emptyset = T,$$

$$d_t s = T \text{ if } t \in s, \text{ otherwise } d_t s = \emptyset.$$

**Proposition 1.6.** *The basic psP-algebra is functionally complete, that is any n-argument operation* $f\colon ET^n \rightarrow ET$, $n = 0, 1, \dots,$ *is definable with the operations of this algebra.*

Given a Boolean algebra $\mathbf{B} = (B, \cup, \cap, \rightarrow, \neg, \mathbf{1}_B, \mathbf{0}_B)$ and a poset $(T, \leqslant)$ by a descending $T$-sequence of elements of $B$ we mean an indexed family $(b_t)_{t\in T}$ of elements of $B$ such that $w \leqslant t$ in $T$ implies $b_t \leqslant b_w$ in $B$ (for the sake of simplicity we denote the Boolean ordering of $B$ with the same symbol). We say that $B$ and $T$ satisfy condition (erpc) of existence of relative pseudo-complement if

(erpc) For any two descending $T$-sequences $b = (b_t)_{t\in T}$, $c = (c_t)_{t\in T}$ of elements of $B$ there exists $\bigcap^B\{b_w \rightarrow c_w : w \leqslant t\}$ for all $t \in T$.

*Example* 1.2. We present a psP-algebra $\mathbf{P}_T(\mathbf{B})$ of type $T$ determined by a Boolean algebra $\mathbf{B} = (B, \cup, \cap, \rightarrow, -, \mathbf{1}_B, \mathbf{0}_B)$ such that $\mathbf{B}$ and $T$ satisfy condition (erpc). The universe $P(B)$ of $\mathbf{P}_T(\mathbf{B})$ is the set of all descending $T$-sequences of elements of $B$. We define a partial ordering $\leqslant$ on $P(B)$ as follows. Let $b = (b_t)_{t\in T}$ and $c = (c_t)_{t\in T}$ be any elements of $P(B)$. Then

$$b \leqslant c \text{ in } P(B) \text{ iff } b_t \leqslant c_t \text{ in } B \text{ for all } t \in T.$$

The system $(P(B), \leqslant)$ is a lattice with join and meet defined by

$$b \cup c = (b_t \cup c_t)_{t\in T}, \quad b \cap c = (b_t \cap c_t)_{t\in T}.$$

Since $B$ and $T$ satisfy (erpc), for any $b, c$ in $P(B)$ there exists the relative pseudo-complement $b \rightarrow c$ and

$$b \rightarrow c = (x_t)_{t\in T}, \text{ where } x_t = \bigcap\{b_w \rightarrow c_w : w \leqslant t\}.$$

For every $s \in ET$ we define

$$e_s = (x_t)_{t\in T}, \text{ where } x_t = \mathbf{1}_B \text{ if } t \in s, \text{ otherwise } x_t = \mathbf{0}_B.$$

Moreover, we put

$$d_w b = (x_t)_{t\in T}, \text{ where } x_t = b_w \text{ for every } t \in T,$$

$$\neg b = (x_t)_{t\in T}, \text{ where } x_t = \bigcap\{-b_w : w \leqslant t\}.$$

It is easy to verify that the algebra

$$\mathbf{P}_T(\mathbf{B}) = (P(B), \cup, \cap, \to, \neg, \{d_t : t \in T\}, \{e_s : s \in ET\})$$

defined above is a psP-algebra of type $(T, \leqslant)$.

**Proposition 1.7.** *Let a Boolean algebra* $\mathbf{B}$ *and a poset* $(T, \leqslant)$ *satisfying (erpc) be given. Let* $\mathbf{P}$ *be the algebra* $\mathbf{P}_T(\mathbf{B})$ *defined as in Example* 1.2*. Then the algebra* $\mathbf{B}_P$ *(see Proposition* 1.4*) is isomorphic to* $\mathbf{B}$*.*

*Example* 1.3. A particular instance of the algebra defined in Example 1.2 is a set algebra obtained by taking the field of all subsets of a set as the respective Boolean algebra. Let $U$ be a nonempty set and let $B(U)$ be the field of all subsets of $U$. We have $\mathbf{1}_{B(U)} = U$ and $\mathbf{0}_{B(U)} = \emptyset$. For any poset $(T, \leqslant)$, $B(U)$ and $T$ satisfy condition (erpc). Let $P(B(U))$ be the set of all descending $T$-sequences of sets from $B(U)$. The ordering on $P(B(U))$ is the set inclusion. The algebra $\mathbf{P}_T(\mathbf{B}(U)) = (P(B(U)), \cup, \cap, \to, \neg, \{d_t : t \in T\}, \{e_s : s \in ET\})$ defined as in Example 1.2 is a psP-algebra of type $(T, \leqslant)$. The infinite joins in the axiom (p6) are set unions.

**Proposition 1.8 (Representation theorem).** *Let* $\mathbf{P} = (P, \cup, \cap, \to, \neg, \{d_t : t \in T\}, \{e_s : s \in ET\})$ *be a psP-algebra of type* $(T, \leqslant)$*. If* $T$ *is denumerable and either well-founded or the set* $M(P)$ *is denumerable (in particular if* $P$ *is denumerable), then for any denumerable set* $Q$ *of infinite joins and meets in* $P$ *there exists the field* $B(U)$ *of all subsets of a nonempty set* $U$ *and a monomorphism* $h$ *from* $\mathbf{P}$ *into* $\mathbf{P}_T(\mathbf{B}(U))$ *preserving all the operations in* $Q$*.*

## 2. Post algebras of order m

The first axiom system for the algebras characterising Post's $m$-valued logics, for a finite m greater than 2, was presented in Rosenbloom [43]. He called them Post algebras. The axiomatisation was then simplified in Epstein [9] and Traczyk [48]. Traczyk proved the equational definability of the class of Post algebras. Over the years the theory of Post algebras and several generalizations of these algebras have been developed. Here we define Post algebras of order $m$ as a particular case of psP-algebras.

Let $(T_m, \leqslant)$ be a poset such that $T_m = \{1, \ldots, m-1\}$, where $m$ is a natural number greater than 2, and $\leqslant$ is a natural ordering in $T_m$. Then $ET_m = \{\emptyset, s(1), \ldots, s(m-1)\}$, where $s(t) = \{w \in T_m : w \leqslant t\}$. Clearly, $(ET_m, \subseteq)$ is isomorphic to $\{0, 1, \ldots, m-1\}$ with the natural ordering. Hence, we can identify these two posets and assume that constants $e_s$ are indexed with elements from $\{0, 1, \ldots, m-1\}$.

By a Post algebra of order $m$ we mean a psP-algebra of type $(T_m, \leqslant)$.

It can be easily shown that this definition is equivalent to the standard definition of Rousseau [44, 45].

*Example* 2.1. A classical example of a Post algebra of order $m$ is an $m$-element Post algebra such that $P = \{e_0, \ldots, e_{m-1}\}$, and for $i, j \in \{0, 1, \ldots, m-1\}$ the operations in $P$ are defined as follows:

(ex1) $e_i \cup e_j = e_{\max(i,j)}$,

(ex2) $e_i \cap e_j = e_{\min(i,j)}$,

(ex3) $e_i \rightarrow e_j = \mathbf{1}$ if $i \leqslant j$, otherwise $e_i \rightarrow e_j = e_j$,

(ex4) $\neg e_i = e_i \rightarrow \mathbf{0}$,

(ex5) $d_i e_j = \mathbf{1}$ if $i \leqslant j$, otherwise $d_i e_j = \mathbf{0}$.

**Proposition 2.1.**

(a) $e_0 \rightarrow a = e_{m-1}$,

(b) $e_s \rightarrow a = \bigcup\{d_t a \cap e_t : t \leqslant s\} \cup d_s a$,

(c) $e_{m-1} \rightarrow a = a$,

(d) $a \rightarrow e_s = e_s \cup \neg d_{s+1} a$, *for* $s = 0, \ldots, m-2$,

(e) $a \rightarrow e_{m-1} = e_{m-1}$.

We define disjoint operations $c_s$ for $s \in \{0, 1, ..., m-1\}$ as follows:

(c1) $c_0 a = \neg d_1 a = \neg a$,

(c2) $c_s a = d_s a \cap \neg d_{s+1} a$ for $s \in T \setminus \{m-1\}$,

(c3) $c_{m-1} a = d_{m-1} a$.

We clearly have

$$c_s a \cap c_t a = e_o \text{ for } s \neq t.$$

Any element $a$ of $P$ has the following disjoint representation:

(c4) $a = \bigcup\{c_t a \cap e_t : t \in T\}$.

Theorems analogous to propositions 1.1, …,1.8 hold and constructions from examples 1.2 and 1.3 carry over to the case of type $(T_m, \leqslant)$. Algebras presented in Example 1.1 can be identified with those defined in Example 2.1. Moreover, the algebras $\mathbf{B}_P$ and $\mathbf{C}_P$ corresponding to a Post algebra of order $m$ coincide.

The $m$-valued Post logic is a propositional logic with binary connectives $\wedge$, $\vee$, $\rightarrow$, unary connectives $\neg$, $D_t$ for $t \in T$, and propositional constants $E_s$ for $s \in \{0, 1, \ldots, m-l\}$. The algebraic semantics for the logic is determined in the standard way by the class of Post algebras of order $m$. A Hilbert-style axiomatisation of $m$-valued Post logic and its completeness with respect to the algebraic semantics is presented in Rasiowa [30]. The main results on $m$-valued Post logic include: Model existence theorem (Rasiowa [31]), Craig interpolation theorem (Rasiowa

[31]), Herbrand theorem (Perkowska [23]). Applications of the $m$-valued Post logic are concerned with the theory of programming. An algorithmic logic based on $m$-valued Post logic is developed in Perkowska [23].

Post algebras and logics of any finite type $(T, \leqslant)$ are considered in Nour [16]. They are also treated in Konikowska, Morgan and Orłowska [13].

## 3. Post algebras of order $\omega^+$

Let $(T_\omega, \leqslant)$ be a poset such that $T_\omega = \omega$ is the set of natural numbers and $\leqslant$ is the natural ordering of natural numbers. Then $ET_\omega = \{\emptyset, s(1), s(2), \ldots, T_\omega\}$. Clearly, $(ET_\omega, \subseteq)$ is isomorphic to $\{0, 1, 2, \ldots, \omega\}$ with the natural ordering. Hence, we can identify these two posets and assume that constants $e_s$ are indexed with elements from $\{0, 1, 2, \ldots, \omega\}$.

A Post algebra of order $\omega^+$ is a psP-algebra of type $(T_\omega, \leqslant)$.

An example of a Post algebra of order $\omega^+$ can be defined in a way similar to that developed in Example 2.1.

Theorems analogous to propositions 1.1, …, 1.8 hold and constructions from examples 1.2 and 1.3 carry over to the case of type $(T_\omega, \leqslant)$. Moreover, the algebras $\mathbf{B}_P$ and $\mathbf{C}_P$ corresponding to a Post algebra of order $\omega^+$ coincide.

Representation theory for Post algebras of order $\omega^+$ has been also developed in Maksimova and Vakarelov [15], Rasiowa [39].

A Hilbert-style axiomatisation of $\omega^+$-valued Post predicate logic and its completeness with respect to the algebraic semantics is presented in Rasiowa [33]. The other results on $\omega^+$-valued Post logic include: Kripke style semantics (Maksimova and Vakarelov [14], Vakarelov [52]), Herbrand theorem and a resolution-style proof system (Orłowska [17–19]), relational semantics and a relational proof system (Orłowska [21]).

Applications of the $\omega^+$-valued Post logic are concerned with the theory of programming. An algorithmic logic based on $\omega^+$-valued Post logic is developed and investigated in Rasiowa [36–38].

## 4. Post algebras of order $\omega + \omega^*$

These algebras are introduced and investigated in Epstein and Rasiowa [11, 12]. Let $T = \{1, 2, \ldots, -2, -1\}$ and $E = \{0, 1, 2, \ldots, -2, -1\}$. A Post algebra of order $\omega + \omega^*$ is an algebra of the form (P) in Section 1 satisfying axioms (p0), …, (p6), where in (p0) $\mathbf{0} = e_0$ and $\mathbf{1} = e_{-1}$, and the following

(p7) $d_1 a = d_{-1} a \cup \bigcup\{d_s a \cap \neg d_{s+1} a : 1 \leqslant s \leqslant -1\}$ pivot elimination axiom

(p8) $(a \to b) \cup (b \to a) = 1$.

The axiom (p7) says that an element $e$ such that $e_t \leqslant e$ $(d_t e = \mathbf{1})$ for all positive $t$ and $e < e_t$ $(d_t e = 0)$ for all negative $t$ does not exist.

Propositions analogous to Propositions 1.1, 1.2, 1.3, 1.4 hold for Post algebras of order $\omega+\omega^*$. Moreover, the algebras $\mathbf{B}_P$ and $\mathbf{C}_P$ corresponding to a Post algebra of order $\omega+\omega^*$ coincide.

*Example* 4.1. A most natural example of a Post algebra of order $\omega+\omega^*$ is a linear Post algebra of order $\omega+\omega^*$ defined as follows:

> $P = \{e_s : s \in E\}$, and the operations in $P$ are defined with conditions analogous to (ex1), ...,(ex5) from Example 2.1.

Disjoint operations in Post algebras of order $\omega+\omega^*$ can be defined with conditions analogous to (cl), (c2), (c3) from Section 2 by replacing $m-1$ with $-1$. Then any element $a$ of $P$ has a disjoint representation given by condition (c4).

In Post algebras of order $\omega+\omega^*$ one can define arithmetic-like operations in the following way.

The successor $sa$ (the predecessor $pa$) of an element $a$ of $P$ is an element given by the following disjoint representation

$$\begin{aligned} sa &= \bigcup\{c_t a \cap e_{t+1} : t \in E\} \\ pa &= \bigcup\{c_t a \cap e_{t-1} : t \in E\} \end{aligned}$$

provided that either of these exist.

The inverse $-a$ of an element $a$ is given by the disjoint representation

$$-a = \bigcup\{c_t a \cap e_{-1} : t \in E\}$$

provided that it exists.

Addition and multiplication operations have disjoint representations as follows

$$a+b = \bigcup\{c_t(a+b) \cap e_t : t \in E\}$$

where for each $t \in T$ the infinite join $c_t(a+b) = \bigcup\{c_i a \cap c_j b : i+j=t\}$ exists,

$$a \cdot b = \bigcup\{c_t(a \cdot b) \cap e_t : t \in E\}$$

where for each $t$ there is the finite join $c_t(a \cdot b) = \bigcup\{c_i a \cap c_j b : ij = t\}$.

**Proposition 4.1.** *A Post algebra of order $\omega+\omega^*$ with inverse, addition and multiplication is a commutative ring with unit, where the ring zero is $e_0$ and the ring unit is $e_1$.*

These rings have the characteristic 0.

For a descending $T$-sequence $X = (X_t)_{t \in T}$ of sets from the field $B(U)$ of all subsets of a nonempty set $U$ we define

$$\begin{aligned} X^+ &= \bigcap\{X_t : t \text{ positive}\} \\ X^- &= \bigcup\{X_t : t \text{ negative}\}. \end{aligned}$$

It can be shown that the algebra of descending $T$-sequences $X = (X_t)_{t \in T}$ of sets from $B(U)$ such that $X^+ = X^-$, with the operations defined as in Example 1.2, is a Post algebra of order $\omega + \omega^*$.

Representation theorem Post algebras of order $\omega + \omega^*$ has the following form.

**Proposition 4.2 (Representation theorem).** *For every denumerable Post algebra* $\mathbf{P}$ *of order* $\omega + \omega^*$ *there is a monomorphism* $h$ *of* $\mathbf{P}$ *into a Post set algebra of order* $\omega + \omega^*$ *whose elements are descending* $T$*-sequences* $X = (X_t)_{t \in T}$ *of sets from the field* $B\{U\}$ *of all subsets of a nonempty set* $U$ *such that* $X^+ = X^-$. *Moreover,* $h$ *preserves a given denumerable set* $Q$ *of infinite joins and meets of* $\mathbf{P}$.

Applications of the logic are concerned with approximation reasoning. An approximation reasoning to recognise a subset $S$ of a nonempty universe $U$ is understood as a process of gradual approximating $S$ by

subsets of $U$ $S \subseteq S_1 \subseteq S_2 \subseteq \ldots$ which cover $S$

and subsets $\ldots \subseteq S_{-2} \subseteq S_{-1} \subseteq S$ which are contained in $S$.

Then the approximations of set $S$ are defined as follows:

$$\begin{aligned} S^+ &= \bigcap\{S_t : t \text{ positive}\} \\ S^- &= \bigcup\{S_t : t \text{ negative}\}. \end{aligned}$$

In Epstein and Rasiowa [12] a characterisation of sets $S$ such that $S^+ = S^-$ is given.

Post algebras of order $\vartheta$, where $\vartheta$ is an arbitrary ordinal number are introduced and investigated in Przymusinska [26–29].

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