All Quantifiers versus the Quantifier All

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Abstract In courses of logic for general students the general and existential quantifiers are the only ones distinguished from among all possible quantifier expressions of the natural language. One can argue that other quantifiers deserve mention, even though there are good reason for emphasizing the familiar ones: namely, they are the simplest, the universal quantifier is a counterpart of the operation of generalizing, the number of nested quantifiers is a good measure of logical complexity, and the expressive power of the general quantifier and its dual is considerable.

Yet, even in the teaching about these two simplest quantifiers it has not been resolved how to indicate the realm to which a given quantifier refers. The methods range from the Fregean assumption that they refer to the totality of objects in the world to the restricted quantifiers to many sorted logic. It turns out that these approaches are not fully equivalent, because the sorts are usually assumed to be nonempty, which results in a problem similar to the well-known issue with non-emptiness of names in syllogistics.

Logicians have studied various generalized quantifiers. It is, however, unclear how to treat the quantifier "many" and similar heavily context-dependent ones. They are not invariant under isomorphisms so no purely logical or mathematical treatment seems applicable. How else can one characterize the context-independent quantifiers among all possible quantifiers corresponding to quantifier expressions in natural language? The following thesis on quantifiers is proposed:

(Principal Thesis) Context-independence = definability in terms of the universal quantifier.

This thesis provides an additional reason for distinguishing the universal quantifier from among all other quantifiers: it suffices for defining all context-independent ones.

Keywords Quantifier, generalized quantifiers, quantifier's range, definability, context-independence

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1 The universal quantifier and its dual

The introduction of quantifiers to logical systems constituted an essential progress with respect to the calculus of Boolean connectives – even though the propositional calculus was finally formulated as a system at about the same time as was the predicate (functional) calculus. The wider system, including the quantifiers "for all" ($\forall$) and "there exists" ($\exists$), was also a far reaching strengthening of syllogistics, the celebrated system that up to the 19th century was seen as the core of logic, and that can be seen as an ancient form of a fragment of the predicate calculus.

In logic, from Aristotle to Frege to mid-20th century predicate logic, only two quantifiers were incorporated: the general and the existential. They are still the only ones taught in general logic courses. Because in classical logic $\exists$ is the dual of $\forall$, that is, $\exists = \neg\forall\neg$, we can say that only the universal quantifier is added. (In some non-classical logic, e.g., the intuitionistic logic, we need to retain the two quantifiers.)

From a certain natural point of view, namely the approach based on the linguistic realities, it is not clear why the general and existential quantifiers are the only concepts distinguished from among all possible quantifier expressions of the natural language. In natural language there are dozens quantifier expressions, that is, expressions that state or estimate the number of objects of a certain kind, or the size of a collection, or compare sizes, etc. They include phrases like ‘all’, ‘always’, ‘nowhere’, ‘almost never’, ‘most’, ‘infinitely many’, ‘many’, ‘from time to time’, ‘a few’, ‘quite a few”, ‘several’, ‘just one’, ‘at least one’, ‘an overwhelming part of’, ‘as many as’, ‘roughly as many as’, and many, many more, including statements like “More girls study programming than boys learn boxing.” In mathematics, some other quantifier expressions are used, for example ‘there are finitely many’, ‘there are uncountably many’, ‘the set of … is dense in …’, and the phrases like ‘almost all’, ‘a negligible amount’ are given various precise meanings in specific mathematical theories.

For a long time logic did not recognize the rich realm of quantifiers or at least did not perceive it as belonging to the domain of logic. What could be the reason for the distinguished role of the familiar quantifiers? Let us try to argue from a logical perspective.

First, simplicity. ‘All things’ corresponds to the full set – either of all things or of all things in our universe of discourse. ‘At least one thing’ corresponds to the notion of non-empty set, or is the negation of being the empty set.

The two standard quantifiers are the simplest ones. At the same time, and this is the second reason, we can see the general quantifier as an abstract counterpart of the operation of generalization, our important mental faculty. (Existential quantifier is, as mentioned above, its dual.) This operation can be seen as basic: in Richard Epstein’s *Critical thinking* [1] only the operation of generalization is mentioned, the quantifiers are not.

The third and fourth reasons are given after the following Digression and a fifth reason emerges at the end of the paper.

2 A digression on the range of the familiar quantifiers

Whereas the quantifiers $\forall$ and $\exists$ are familiar now, their use causes some problems that are not trivial, especially in the educational context, when general students are taught. One problem is: what is the range of quantifiers, the realm to which a given quantifier refers? Another problem appears when a standard answer is given to the first one: how to indicate the range?

Frege introduced quantifiers in his framework, in which everything, literally: *every thing*, was included. For Frege, and similarly for his successors like Russell, (early) Wittgenstein or Quine, there is only one value-range for quantifiers, namely “all the actually existing individuals” (cf. Hintikka [7, p. 30] and Peters-Westerståhl [17, p. 40]) or even “all the conceivably existing individuals” (as in Russell [18]). That assumption seems, however, unsatisfactory. There are at least two main reasons for dissatisfaction. First, it seems that in order to apply this approach the world must be perceived as a collection of things. In particular, ‘always’ is expressible by $\forall$ only if time is seen as composed of things such as moments or segments. In addition, only timeless relations are naturally dealt with (cf. Epstein [2] and [3], Appendix A). Second, to generalize over everything seems odd. In practice, we almost always mean a specified limited range. Today we rarely share Frege’s ontology, but we all continue to use his formalism (in a modified form, of course). The contemporary prevailing approach to this formalism is, however, vastly different from his. The world is complex, and, usually, in a given moment we consider only some objects, that is, we specify fragments of the whole world. We have overwhelmingly adopted the model theoretic approach: models vary, and generality means "for all elements of an intended range." The formalism remains but its interpretation is different: logic is no more about "the world" but rather about various "possible worlds", or models. It was Tarski who helped convince logicians to study truth in models. In Hintikka [7] it is also stressed that this new, model-theoretic approach overcomes the difficulty inherent in Frege's approach, namely, how to identify the basic simple things, the "urindividuals".

Having agreed on the limited range, we need to express that in the symbolism we use to deal with quantifiers. There are two traditional ways of expressing the restriction on the range of variables used inside (logical) formalism. One is the use of variables of different sorts. The other is the use of restricted, or relativized, quantifiers. As is well known, it is easy to express relativized quantifiers by unrestricted ones: for any predicates A, B, C

$$(\forall x)_{C(x)} A(x) \equiv (\forall x)(C(x) \to A(x)), \quad (\exists x)_{C(x)} A(x) \equiv (\exists x)(C(x) \wedge A(x)).$$

In terms of a variable t, assumed to satisfy $C(t)$, we simply have $(\forall t)A(t)$ and $(\exists t)A(t)$.

Is there a difference between the two methods? Of course, the approach is different: the restrictions can be seen as imposed from outside the system in the case of the language with different sorts of variables, while they are an optional part within the system in the case of relativisation. Still, at the first glance it may seem that they are formally equivalent. But not quite – there is a subtle difference. We normally assume that all sorts are nonempty – similarly to the assumption that the universe is nonempty, or that in each model the universe of the model is nonempty. This assumption is not made about the predicate $C(x)$.

Sometimes the possible emptiness of C is harmless. Explicit restrictions on the quantifiers preserve the validity of de Morgan's laws:

$$\neg(\forall x)_{C(x)} A(x) \equiv (\exists x)_{C(x)} \neg A(x), \qquad \neg(\exists x)_{C(x)} A(x) \equiv (\forall x)_{C(x)} \neg A(x).$$

What about the other tautologies that are so useful in manipulating quantifier prefixes? Using different sorts brings no harm. In contrast to that, relativisation may cause a problem! It seems to me that while the story with (some) Aristotelian syllogisms being valid only under the assumption that all the terms are non-empty is very well known, similar limitations concerning relativized quantifiers are not generally known. It is easy to see (as was remarked in Krajewski [11]) that the following theorem holds.

Remark on relativized quantifiers:

The following formulas are valid under relativisation to an arbitrary C (we assume that in A the variable x is not free):

$(\forall x)(A \vee B(x)) \equiv A \vee (\forall x)B(x)$,

$(\forall x)(A \to B(x)) \equiv A \to (\forall x)B(x)$,

$(\exists x)(A \wedge B(x)) \equiv A \wedge (\exists x)B(x)$,

$(\forall x)(B(x) \to A) \equiv (\exists x)B(x) \to A$,

The following formulas remain valid only when relativized to nonempty C (we assume that in A the variable x is not free):

$(\forall x)(A \wedge B(x)) \equiv A \wedge (\forall x)B(x)$,

$(\exists x)(A \vee B(x)) \equiv A \vee (\exists x)B(x)$,

$(\exists x)(A \to B(x)) \equiv A \to (\exists x)B(x),$

$(\exists x)(B(x) \to A) \equiv (\forall x)B(x) \to A.$

In other words the relativisations of the above tautologies are not valid, they are sometimes false when C is interpreted as an empty set, but the following formulas are valid:

$(\exists x)C(x) \to [(\forall x)_{C(x)}(A \wedge B(x)) \equiv A \wedge (\forall x)_{C(x)}B(x)],$

$(\exists x)C(x) \to [(\exists x)_{C(x)} (A \vee B(x)) \equiv A \vee (\exists x)_{C(x)} B(x)],$

$(\exists x)\ C(x) \to [(\exists x)(A \to B(x)) \equiv A \to (\exists x)B(x)],$

$(\exists x)C(x) \to [(\exists x)_{C(x)} (B(x) \to A) \equiv (\forall x)_{C(x)} B(x) \to A].$

3 The power of ∀ and ∃

The third reason for the distinguishing of ∀ and ∃ from among all possible quantifiers has to do with logical complexity. The number of nested quantifiers is a good indicator of logical complexity. The quantifiers ∀ and ∃ provide a great measure of complexity if the number of alternating nested quantifiers is counted. The realization of this possibility gave rise to the Kleene-Mostowski hierarchy, classifying the sets obtained from recursive sets by a series of projections and complements. (See Kleene [10], Mostowski [15].) Then other similar growing chains of ever more complicated objects were established, e.g., the analytic hierarchy. From such a perspective these simple familiar quantifiers look like anything but trivial. It is also of interest that neither Aristotle nor other pre-modern logicians considered nested quantifiers. If teaching about quantifiers is limited to formulas, an especially tautologies, with one quantifier or at most two, as is still done in coursed for general students, the matter looks rather trivial and it remains unclear why the quantifiers are needed. The power of quantifiers, even the simplest ones, is seen only when several are combined. This brings us to the next reason for the distinguishing of ∀ and ∃.

The fourth reason emerges when one realizes that these quantifiers bring much more expressive power than it would seem at first. When the standard additional machinery available in logic is employed many new quantifiers can be defined. Some of them can be easily defined within first order logic, for instance the numerical quantifiers: "there are exactly n", in short $\exists^{!n}$, "there are more than n", in short $\exists^{>n}$, and their combinations (like "there are three or four"), etc.

In higher order logics and in set theory many more quantifiers can be defined. Definitions in mathematics are expressed in a technical language of a given branch, but logicians have been able to express these definitions in the language of logic. Thus, for instance, "there are infinitely many" cannot be defined in the 1st order logic, but can be defined in the 2nd order logic. The Henkin quantifier, the first example of a branching quantifier, namely "for every x there exists y, and independently of that for every z there exists t such that $R(x, y, z, t)$", also goes beyond 1st order logic (see Henkin [6] and Krynicki et al. [13]), even though it reflects such a way of using the familiar quantifier expressions corresponding to ∀ and ∃ that can be found in natural language; this quantifier is easily defined in 2nd order logic: "there exist functions f, g such that for every x and for every z $R(x, f(x), z, g(z))$". The phrase "there are uncountably many" also defines a quantifier but it makes sense only in reference to a background set theory. It was unexpected that this quantifier can be recursively axiomatized. (See Keisler [8].) There are many more examples of mathematical quantifiers. They suggested to mathematical logicians the concept of a "generalized quantifier".

4 Generalized quantifiers in logic

Generalized quantifiers were introduced to logic by Mostowski [16]. The formula $(Qx)\varphi(x)$ is satisfied in a model $\mathrm{M} = (M,\ldots)$ iff the set $\{a\colon \mathrm{M} \models \varphi[a]\}$ belongs to the family of subsets of M that serves as the interpretation of Q. (Thus $\forall$ is interpreted as $\{M\}$ and $\exists$ as the family of all non-empty subsets of M.)

This notion was useful but was not sufficient for many formulations that are used in natural language. Mostowski quantifiers are all of type <1>. The sentence "More girls study programming than boys learn boxing" cannot be analyzed in logic with type <1> quantifiers only. A more general definition was introduced by Lindström [14] who allowed quantifiers of an arbitrary type $<n_1,\ldots,n_k>$ that bind more variables and apply to several formulas, and in a model M are interpreted as relations between subsets of M (in the case of monadic quantifiers of type <1,1,…,1>) or, more generally, relations between relations on M.

Both Mostowski and Lindström were mathematicians so they made an important assumption which obviously seemed necessary to them: they consider only the quantifiers that are invariant with respect to isomorphism. Formally, if $\mathrm{M} \cong \mathrm{M}'$ then

$$\mathrm{M} \models (Qx_1,\ldots x_k)(\varphi_1,\ldots,\varphi_n) \text{ iff } \mathrm{M}' \models (Qx_1,\ldots x_k)(\varphi_1,\ldots,\varphi_n).$$

The assumption in the case of monadic quantifiers amounts to the fact that only the size of the sets defined by the quantified formulas matters (cf. Peters-Westerståhl [17] or Westerståhl [20]). The assumption that logic should be completely topic-neutral constitutes the reason for admitting into logic only the quantifiers invariant under isomorphism. Other mathematical properties can be defined by isomorphism-preserving quantifiers. Yet they are not sufficient for some quantifiers commonly used in natural language.

It is clear that logic is poorly equipped, if at all, to deal with many from among the quantifier expressions listed above. For example, the concept "many" is different from the more logical quantifiers and seems hardly definable in general since its meaning depends on the situation in which the term is used. It is context-dependent. Peters and Westerståhl call it "strongly" context-dependent and some authors call it intensional. (See Peters-Westerståhl [17, p. 213].) To evaluate a sentence with such a context-dependent quantifier we need an appropriate understanding of the world, or at least of the appropriate fragment of the world. Logic itself is not sufficient. To know whether it is true or not that *many* women at my university are pregnant or that *many* have been in Himalayas, we need to know how many women of a given age are, on average, pregnant, and how many go to Himalayas.

It is similarly with quantifier expressions like "a few", "several", "a huge number", "rarely", "often", etc., and even more obviously, with "surprisingly many", "almost everyone", "virtually nowhere", etc.

I believe that it should always be taught in general logic courses that many concepts can be defined by the simplest quantifiers, but at the same time the student should be made aware that many natural language constructions cannot. This seems to be generally ignored by teachers of logic. For example, in the otherwise comprehensive textbook by Andrzej Grzegorczyk [5] the other quantifiers are not even mentioned.

This postulate seems to be loosely connected to a remark by Gödel, possibly his only recorded statement on the present topic. According to Wang ([19, p. 266]), Gödel said: "Even though predicate logic is distinguished there are also other notions, such as *many*, *most*, *some* (in the sense of plurality), and *necessity*."

Despite the initial impression that the quantifier "many" is not definable, one could try to define it formally, or to model it, by adding a variable σ and defining "many" as more numerous than (the interpretation of) σ. This new variable can be either a numerical one, interpreted as a cardinal number, or a set variable, interpreted as a certain set S. Then "many x's (satisfying φ)" is defined as having more members than S, or as the requirement that the cardinality of the set of the values of x that satisfy the interpretation of φ is larger than the cardinality of S. The set S depends on the context; it is chosen specifically for each interpretation.

The problem with this attempt is that the definition of "a few" is the same, only with "<" instead of ">". And the phrase "more than a few" is formalized exactly as is "quite a few" and "many". And do we normally identify "many" with "more than a few"? Hardly.

The above remarks should be easy to understand, but an example can still be helpful. A certain number, say 7, can play the role of both delimitations in the same discourse. For example, if exactly 7 students among the 20 students in my Warsaw university class have read more than ten books in their lifetime and 7 are pregnant, I would say that it is true that "a few read books" and "many are pregnant". (Incidentally, I believe that there could exist schools somewhere in the world in which 7 pregnant among 20 students would be seen as "few", and 7 readers among them would be considered "many".)

So everything depends on the context and introducing σ is of no help. Only the context counts.

We can still maintain that the logical content is better explained, when this formalization is made. The important feature – and a problem from a normal logical perspective – is that "many" defined as "more than σ" is not invariant with respect to ismorphisms. To continue our example, $(G, P) \cong (G, R)$, where G is the class, P is the set of pregnant students in the class, R is the set of book readers in the class, but the sentence "there are many x that φ" is true in one and false in the other interpretation.

What is more, the quantifier "many" does not have some monotonicity properties. It may happen that $M \models [(\forall x)\ (\varphi(x) \rightarrow \psi(x))]$ and still $M \models [(\text{Many } x)\ \varphi(x)\ \&\ \neg(\text{Many } x)\ \psi(x))]$. It may happen even if the inclusion of (the interpretation of) φ in (the interpretation of) ψ is strict. For example, if there were 8 students reading books, including each of the 7 who are pregnant, there would still be many pregnant students and not many readers in the class.

Dealing with context-dependent quantifiers one can wonder how many contexts there are. Infinitely many? This seems probable, at least in the case of a quantifier "many". Is this the reason we are unable to pin them down? If only finitely many contexts were possible, a fixed number, then perhaps we could give a definition by listing all the cases. Notice that if a finite but practically unmanageable number of contexts has to be taken into account then the quantifier is still undefinable by us. However, a sufficiently strong intelligence, or even robot, could perhaps do that. The problem is analogous to the problem whether a computer can handle the natural language. The hopes of some early pioneers of Artificial Intelligence that computers would speak as humans were naive. Yet in restricted settings, where contexts can be comprehensively listed, it is perfectly possible to have computers "speak."

5 Characterizing context-independent quantifiers

It seems that context-independence means that any extralogical terms referring to some specific fragments of the world are irrelevant for the understanding of the formula. The topic covered in the statement is of no consequence, only logic counts. Thus

(1) Context-independent quantifiers = Topic neutral quantifiers

We can still maintain that a definition of (a quantifier) being context-independent is needed. This is clear in specific cases, but can a general definition be given? What is needed is a criterion – indeed, a context-independent criterion – for context-independence of quantifiers (or perhaps even more generally, context-independence as such). The idea is, of course, quite simple: there is no need for any specific knowledge about the world. Yet, one could say, to understand the Magidor-Malitz quantifier one certainly needs some non-trivial knowledge. It is, however, a purely logical knowledge (in the broad meaning of logic), different from the knowledge of the features of the world, physical or social, that are relevant for the specific situation. We might even try to say that what is needed for understanding the context-independent quantifiers is the familiarity with merely the necessary features of the world. One could ignore its contingent aspects.

It has been noticed above that when linguistic quantifier expressions are reconstructed within logic the requirement of context-independence is formulated as invariance with respect to isomorphisms. Thus, we get another thesis:

(2) Context-independent quantifiers = Quantifiers invariant under isomorphisms

Before another thesis proposing a characterization of context-independence of quantifiers is attempted let us consider the meaning of being a thesis in this context. Church's Thesis is the best known example of a thesis identifying a formal concept with an intuitive one. The mathematical concept of recursive function is identified with the intuitive concept of effectively computable function. For a long time, the general conviction was that such a thesis can be justified by various arguments, but there is no way to prove its correctness because the intuitive concept is too vague to be part of a proof. However in recent decades there have been various attempts (in particular by Robin Gandy, Wilfried Sieg, Yuri Gurevich) to prove the identification. Namely, a proper analysis of the intuitive concept of computability can provide principles that make possible a demonstration that a function satisfying them must be recursive. There are more examples of similar theses, for instance "the Cantor-Dedekind thesis" that real numbers are defined by the appropriate set theoretic constructions. (For a discussion of Church's Thesis and the other examples as well as references to literature see, e.g., Krajewski [12].)

In the case studied in the present paper, it is the context-independence applied to quantifiers that is the intuitive notion we want to characterize.

In addition to topic-neutrality and invariance under isomorphisms we can try look at the ways the quantifier can be defined. It seems that whatever definition is formulated it cannot be expressed without taking some specific logic into account. This is because quantifiers are logical objects. They function inside a logical framework. On the other hand, it would be to emphasize the logical nature but ignore any specific logic. The way out of the dilemma can be as follows: the defining property is assumed to make sense in whatever logic it is formulated. For instance, the phrase "$\varphi(x,y)$ defines a well-ordering" defines a type <2> quantifier, whether in 2nd order logic or in set theory.

Any quantifier Q can give rise to a "logic" L(Q). Then Q is trivially definable in this logic. To avoid this triviality, let us call a logic *basic* if it is 1st order, 2nd order, n-th order, type theory or set theory. Hence the following thesis

(3) A quantifier is context-independent iff it is definable in some basic logic.

Because the common part of all such logics, as far as quantification is concerned, is the universal quantifier ∀, we can reformulate the thesis as

(3') A quantifier is context-independent iff it is ∀-definable in some (basic) logic.

Since we admit definability either in 1st order or 2nd order or higher order logic or in (formalized) set theory, and the general quantifier appears in each of these logics we can say in short:

(4) A quantifier is context-independent iff it is definable in terms of ∀,

or briefly,

(Principal Thesis) Context-independence = definability in terms of ∀.

It is seen that the position of the general quantifier, or rather of our two familiar quantifiers, ∀ and ∃, is vindicated. This is the fifth – in addition to simplicity, the faculty of generalization, the measuring of complexity, and the expressive power – and rather unexpected reason for distinguishing ∀: in the presence of the appropriate amount of logical machinery but with no generalized quantifiers ∀ suffices to define all context-independent quantifiers. Thus the power of the universal and existential quantifiers is claimed to be even stronger than it seemed on the basis of the definability of so many quantifiers by ∀. According to the Principal Thesis, the power of ∀, at least in relation to quantifiers, extends to the whole realm of context-independence.

Let us repeat that in each basic logic the universal quantifier is included, so definability in terms of ∀ is really the same as definability in (predicate) logic. The Principal Thesis says that not only definability of generalized quantifiers in terms of ∀ gives context-independence but also that context-independent generalized quantifiers are so definable. Each specific example of a quantifier has been defined (in the proper logic) in terms of ∀; the Thesis states the generalization to all possible quantifiers.

6 Formalism-free definition of ∀- definable quantifiers?

There exists an alternative way of looking at the Principal Thesis. If we agree to it then we can treat the Thesis as the proposal to characterize the definability of quantifiers with the use of ∀ (in basic logics) as context-independence, that is, a feature formulated without the recourse to a specific syntactic machinery used in definitions. This brings to mind the problem of formalism-free characterization of concepts.

The issue of "formalism freeness" has been introduced by Gödel who commented on the fact that all formal definitions of computable functions give the same class of functions. Therefore, even though each definition requires some specific formalism, we have been able to isolate an important class of functions in a formalism-free manner. Computability is formalism-free, and Gödel [4] proposed to look for a similar grasping of definability and other notions. Some developments in mathematical logic, notably work done in model theory by Shelah and Zilber, can be seen as going in this direction – see Kennedy [9]. According to Shelah, in model theory, conceived as a tower, "the higher floors do not have formulas or anything syntactical at all." (Kennedy [9, p. 355])

In our case, the Principal Thesis gives the formalism-free characterization of the class of quantifiers definable in some standard (predicate) logic, that is, using some formalism. In each of these logics we have the quantifier ∀, so one can say that this is the class of quantifiers definable in logic by ∀. The class consists of context-independent quantifiers. This characterization is formalism-free.

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