Foundations and Philosophy of Mathematics in Warsaw, The school of Andrzej Mostowski and philosophy

Marcin Mostowski

Abstract. The paper is a mathematical and philosophical essay devoted to mathematical logic school created and guided by Andrzej Mostowski. Firstly, We discuss some of the main – still actual – achievements of Andrzej Mostowski, then we discuss weak sides of his scientific project. They are computational and philosophical.

Scientific challenges of our times in logic are mainly computational. The warsaw school of mathematical logic did not support this direction.

Partially because of political situation in Poland public philosophical discussions were strongly influenced by hard–politics, including personal politics in academic institutions. Therefore many people thinks that isolation of foundations and philosophy was forced by communist ideology. The impression is false. The abyss between philosophy and foundations was basically independent of political situation in Poland.

What we can learn from this experience?

00A30, 03A05, 00A35, 03E25, 03C80, 03F30

Andrzej Mostowski, Axiom of Choice, Skolem arithmetic, generalized quantifiers, foundations of mathematics, philosophy of mathematics

Introduction

This work can be divided into two parts. The first one is devoted to three selected topics of Andrzej Mostowski work in logic. Undoubtedly, it could not cover all his important and influential works by him. The selection is partially personal and partially on my opinion what was characteristic to his scientific interests.

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The second part is a philosophical and historical essay devoted to relations of his school to philosophy and a new way of computational way of thinking. In comparison to the first part it is much more personal and informal.

1. Some mathematical works in the foundations

We discuss here some selected ideas of Andrzej Mostowski.[1]

1.1. Permutation models

One of the most fascinating idea in foundation of mathematics of XX century was undoubtedly the Axiom of Choice (AC).[2] AC says that for every family F of nonempty sets there is a choice function $f : F \longrightarrow \bigcup F$ such that $\forall A \in F \; f(A) \in A$. Zermelo invented it trying to prove his theorem, TZ: Every set can be well ordered. He proved TZ assuming AC. However, having TZ it is easy to prove AC. For any family F of nonempty sets, if we take any well ordering R on $\bigcup F$ then we can define a choice function taking $f(A) =$ the R–smallest element of A, for all $A \in F$.

The first question related to AC was whether it can be proved in Zermelo–Fraenkel set theory (ZF). A partial answer was known relatively early. However it was done only for the theory ZF extended by allowing elements not being sets, so called atoms, it is called Zermelo–Fraenkel set theory with atoms (ZFA).[3] Models of ZFA are of the form $V(A) = \bigcup_{\alpha} V_{\alpha}(A)$, where $V_0(A) = A$ and for all ordinal numbers α we take $V_{\alpha+1}(A) = P(V_{\alpha}(A)) \cup V_{\alpha}(A)$, for a limit ordinal λ we take $V_{\lambda}(A) = \bigcup_{\alpha<\lambda} V_{\alpha}(A)$. We assume that the set A is infinite.

Let us consider any permutation σ of the set A, that is a bijection $\sigma : A \longrightarrow A$. It can be extended on all sets from the model by taking $B^{\sigma} = \{x^{\sigma} : x \in B\}$. We say that B is stable for σ if $B^{\sigma} = B$. For any set $Z \subseteq A$ we say that a permutation σ fixes Z if $\sigma(a) = a$, for every $a \in Z$.

Now we define $S(A)$ a subuniverse of $V(A)$ taking all sets B from $V(A)$ such that there is a finite $Z \subseteq A$ such that for every permutation σ of the set A which fixes Z, B is stable for σ. Of course all pure sets, having no atoms in their transitive closures, are stable for any permutation of A, then all of them belong to $S(A)$. Additionally all finite and co–finite subsets of A belong to $S(A)$. On the other hand no infinite and co–infinite $B \subseteq A$ belong to $S(A)$. Because for any finite $Z \subseteq A$ we can find $a, b \in A - Z$ such that $a \in B$ and $b \notin B$, then we take a transposition σ exchanging a and b and not moving anything else. σ fixes Z and $B^{\sigma} \neq B$.

The next step is proving that $S(A)$ is a model for ZFA provided $V(A)$ is a model for ZFA. The class $S(A)$ would not contain any well ordering of A. Otherwise, having a well ordering R on A we can split A into two disjoint sets B

[1]Some of the other topics are discussed in part 4 of this book.
[2]A very good monography of the topic can be found in [6].
[3]A good introduction to ZFA can be found in [5].

containing even successors in the sense R and the remaining part containing odd successors.

In this way we gave a sketchy proof of the following:

Theorem 1. *If ZFA is consistent then neither AC nor TZ are provable in ZFA.*

By refinement of the method of Fraenkel, Andrzej Mostowski proved the following:

Theorem 2 ([10]). *In Zermelo–Fraenkel set theory with atoms (ZFA) Zermelo theorem, saying that every set can be well ordered, is independent of Ordering Principle, saying that every set can be linearly ordered.*

Later on he elaborated the method. His book [12] presents the results. Nowadays the basic construction is nowadays called *Fraenkel–Mostowski permutation models.*

These prewar ideas strongly influenced later research in the school of Andrzej Mostowski for many years.

1.2. Skolem arithmetic and direct products of theories

In 1952 Andrzej Mostowski published the paper "On direct products of theories" reprinted in [14]. It contains the first published proof of the theorem commonly attributed to Thoralf Skolem, saying that first order arithmetic of multiplication is complete and decidable.

Neither the idea nor the details of the proof are well known. In 1929 Mojżesz Presburger published his paper presenting his proof of decidability of first order arithmetic of addition [17]. The proof can be found in almost every handbook of mathematical logic.[4] It is one of the paradigmatic proofs by elimination of quantifiers.

The structure $(P^{<\omega}(N), \cup, -, \emptyset)$ of finite sets of natural numbers with union and difference has much more simpler first order theory $T_I = Th(P^{<\omega}(N), \cup, -, \emptyset)$. As a matter of fact this theory is simply the theory of nonprincipal maximal ideals in atomic infinite boolean algebras.[5]

Well known axioms of first order Peano arithmetic PA, with $0, S, +, \times$ as primitive notions, are the following:

(PA1)$\forall x\ (x = 0 \equiv \neg \exists y\ x = S(y)$,
(PA2)$\forall x, y\ (S(x) = S(y) \Rightarrow x = y)$,
(PA3)$\forall x\ x + 0 = x$,
(PA4)$\forall x, y\ x + S(y) = S(x + y)$,
(PA5)$\forall x\ x \times 0 = 0$,
(PA6)$\forall x, y\ x \times S(y) = (x \times y) + x$

[4]Professor Andrzej Grzegorczyk at least twice told me that Alfred Tarski decided that the result by Presbuger was too weak for PhD. Undoubtedly intending this terrible mistake as a lesson for future supervisors.

[5]A proper boolean algebra can be obtained by adding all the complements of the elements of an ideal.

and the induction axiom scheme, for each arithmetical formula $\varphi(x_1, \ldots, x_n, y)$:

(PA7$_\varphi$)$\forall x_1, \ldots, x_n \quad (\varphi(x_1, \ldots, x_n, 0) \wedge \quad \forall y \quad (\varphi(x_1, \ldots, x_n, y) \Rightarrow \varphi(x_1, \ldots, x_n, S(y))) \Rightarrow \forall y\ \varphi(x_1, \ldots, x_n, y))$.

The arithmetic of addition, called also Presburger arithmetic, T_P can be axiomatized by **(PA1)** — **(PA4)** and **(PA7$_\varphi$)** restricted to formulae with $0, S, +$ as only primitive notions.

The arithmetic of multiplication, called also Skolem arithmetic T_S, cannot be so easily extracted from axioms of PA. However it can be defined as the set of all first order consequences of PA which contain only multiplication and all quantifiers are of the form $\forall x \neq 0$ and $\exists x \neq 0$[6] On the other hand, we define T_M as the first order theory of the structure $(N, \times)$. Of course $T_S \subseteq T_M$.

Theories T_I and T_P allow elimination of quantifiers. In the paper "On direct products of theories" Andrzej Mostowski proves that this two methods give elimination of quantifiers for T_M. His theorem is essentially more general. It gives elimination of quantifiers for T_S as a corollary. It was generalized in the work [4].

However we are interested mainly in properties of T_S. The method was elaborated for this case by Patrick Cegielski [1]. Cegielski gives also an axiomatic characterization of Skolem arithmetic, see also [16]. However the argument is still complicated.

The idea of the proof is based on so called Prime Factorization Theorem which says that for each integer $a > 0$ there are uniquely determined a set of prime divisors of a: $Supp(a) = \{q_1 < q_2 < \ldots < q_n\}$ and a sequence of positive integers $a_1, a_2, \ldots, a_n$ such that

$$a = q_1^{a_1} q_2^{a_2} \ldots q_n^{a_n}.$$

The result of multiplication of a by

$$b = r_1^{b_1} r_2^{b_2} \ldots r_m^{b_m}$$

is

$$c = s_1^{c_1} s_2^{c_2} \ldots s_k^{c_k},$$

where $Supp(b) = \{r_1 < r_2 < \ldots < r_m\}$, $Supp(c) = \{s_1 < s_2 < \ldots < s_k\}$, $Supp(c) = Supp(a) \cup Supp(b)$ and for $i = 1, 2, \ldots, k$ the exponent c_i is either the sum of exponents $a_{i'} + b_{i''}$, if $s_i = q_{i'} = r_{i''}$, or c_i is one of $a_{i'}$ or $b_{i''}$, if s_i divides only one of a, b.

Therefore each model $M = (U, \times^M)$ of T_M can be split into the model $(I^M, \cap, -, \emptyset)$ for T_I and the family of models $(M_p)_{p \in Primes^M}$ for T_P, where $I^M = \{Supp(a) \subseteq Primes^M : a \in U\}$ and $M_p = \{Component(p, a) : a \in U\}$, $Component(p, a)$ is the greatest power of the prime p dividing a.

Reception of the proof of Andrzej Mostowski and later refinements is very poor and in what follows we will discuss easier and less general argument for

[6]Zero element in T_S is easy to define and inessential from the point of view of characterizing models of this theory.

completeness and decidability of T_S. Because it is an axiomatic theory then it suffices to prove its completeness.

Much easier proof was given by Nadel [15], who proved the completeness of Skolem arithmetic by using Ehrefeucht–Fraïssé games, [3].[7]

Theorem 3. *Any two models of T_S are elementary equivalent.*

The proof given by Nadel combines winning strategies of $\exists$–playerfor two models of T_I and two models of T_P for obtaining the same for two models of T_S. As a byproduct it gives that $T_M \subseteq T_S$.

This can be done simply by proving in PA proper statements justifying possibility of needed representation. We give two examples of them.

Lemma 1 (Prime Factorization Theorem – a multiplicative version). *In PA the following statement is is provable:*

$$\forall x \forall y\ (x = y \equiv \forall p \in Primes\ \ Component(p, x) = Component(p, y)).$$

Because the statement uses only multiplicative language then it is also provable in T_S.

Lemma 2 (Selection – a multiplicative version). *For each arithmetical first oder formula $\varphi(x_1, \ldots, x_n, y, z)$ The following statement is provable in PA:*

$$\forall x_1, \ldots, x_n\ \forall b\ (\forall p \in Supp(b) \exists a\ ((Pow(p, a) \wedge \varphi(x_1, \ldots, x_n, p, a)) \Rightarrow$$

$$\exists b'\ (Supp(b) = Supp(b') \wedge \forall p \in Supp(b')\ \varphi(x_1, \ldots, x_n, p, Component(p, b')))).$$

If additionally φ is in multiplicative language of T_S then the statement is also provable in T_S.

Proof. Let us fix a formula $\varphi(x_1, \ldots, x_n, y, z)$. Then we prove the statement in PA by induction on the standard enumeration of primes $p_0, p_1, p_2, \ldots$. Firstly we take any $x_1, \ldots, x_n, b$.

We take as b_0 any element such that $Pow(p_0, b_0) \wedge \varphi(x_1, \ldots, x_n, p_0, b_0)$,[8] or $b_0 = 1$ if $\neg\ p_0 \mid b$. Let us assume that $\forall p \in Supp(b) \exists a\ ((Pow(p, a) \wedge \varphi(x_1, \ldots, x_n, p, a))$.

Let us assume that b_n is defined in such a way that $\forall i \leq n \forall p_i \in Supp(b')\ \varphi(x_1, \ldots, x_n, p, Component(p_i, b_n))$.

If $\neg\ p_{n+1} \mid b$ then we take $b_{n+1} = b_n$, otherwise – by the assumption – we have a such that $Pow(p_{n+1}, a) \wedge \varphi(x_1, \ldots, x_n, p_{n+1}, a)$. In this case we take $b_{n+1} = b_n \times a$. We stop at stage n when p_n is the greatest prime in $Supp(b)$, then we take $b' = b_n$. □

[7] The method is currently a standard one and can be found in many textbooks of mathematical logic. A good presentation of it can be found e.g. in [2]

[8] $Pow(q, d)$ means that d is a power of a prime q.

1.3. Generalized Quantifiers

Another very influential work by Andrzej Mostowski was presented in his paper "On a generalization of quantifiers"[11]. The paper does not contain any hard results, but it presents a new very influential idea of *generalized quantifiers*. Traditionally, in mathematics we have used two quantifiers: universal $\forall$ and existential $\exists$. In the Type Theory they have many interpretations in dependency to types of variables bounded. However, restricting our attention to first order – elementary language, they have unique interpretation.

1.3.1. Basic idea. Let us consider a model M with the universe U. Interpretations of quantifiers $\forall$ and $\exists$ depend only on U. So we take

$$\forall_U = \{A \subseteq U : U - A = \emptyset\},$$

and

$$\exists_U = \{A \subseteq U : A \neq \emptyset\}.$$

Now for every formula $\varphi(x)$ with one free variable x we have:

$$M \models Qx\ \varphi(x) \text{ if and only if } \{a \in U : M \models \varphi(a)\} \in Q_U,$$

for any of quantifiers $Q = \forall, \exists$.

Are there more such quantifiers? Restricting our attention only to logical – topic independent – notions, there are much more than two. Logicalness condition for Q says that for any bijection $f : U \longrightarrow W$ and for any $A \subseteq U$:

$$A \in Q_U \text{ if and only if } f(A) \in Q_W,$$

where $f(A) = \{f(a) : a \in A\}$. Quantifiers satisfying this condition are determined by classes of pairs of cardinal numbers, in the following sense $K_Q = \{(card(A), card(U - A)) : A \in Q_U\}$. For instance $K_\exists = \{(\kappa_1, \kappa_2) : \kappa_1 > 0\}$. On the other hand Q_U can be obtained back from K_Q as follows:

$$Q_U = \{A \subseteq U : (card(A), card(U - A)) \in K_Q\}.$$

The basic notion of a *generalized quantifier* is a quantifier in the above sense satisfying the logicalness condition. They are called also *Mostowski quantifiers*

Andrzej Mostowski observed that logics with such quantifiers hardly would be axiomatizable. For instance the quantifier *there are only finitely many* – $\exists^{<\aleph_0}$, defined as

$$\exists^{<\aleph_0}{}_U = \{A \subseteq U : card(A) < \aleph_0\},$$

allows finite axiomatization of the standard model of natural numbers, what means that, by the Tarski undefinability of truth theorem, the set of theorems cannot be arithmetical, therefore it is not axiomatizable.

The axiomatization can be given by **(PA1)** – **(PA6)** and instead of scheme **(PA7$_\varphi$)** we take $\forall x\ \exists^{<\aleph_0} y\ \ y < x$.[9]

[9]The ordering is defined by $x \leq y \equiv \exists z\ x + z = y$ and $x < y$ means $x \leq y \wedge x \neq y$.

Therefore, the result obtained by H. Jerome Keisler [7] giving an example of axiomatizable logic was surprising. He gave a complete axiomatization for the logic with the quantifier $\exists^{>\aleph_0}$, defined as

$$\exists^{>\aleph_0}{}_U = \{A \subseteq U : card(A) > \aleph_0\}.$$

Let us consider the logic $FO(Q)$ first order logic with an additional quantifier Q interpreted as arbitrary Mostowski quantifier. It means that formulae are interpreted in models of the form (M, Q_U), where M is a usual model and U is its universe.

Theorem 4 (Per Lindström, [9]**).** *The set of all universally valid arithmetical formulae in $FO(Q)$ is not arithmetical and therefore not axiomatizable.*

Proof. Let φ be a conjunction of **(PA1)** – **(PA6)** and the statement

$$\psi \;=\; \forall x(Qy\; y < x \equiv \neg Qy\; y < S(x)).$$

Let us observe that ψ says that Q gives different truth values on initial segments determined by a and by $S(a)$, for all a, what is possible only when these segments are finite. Therefore for each arithmetical sentence ξ the conjunction $(\varphi \wedge \xi)$ is consistent in $FO(Q)$ if and only if ξ is true in the standard model of natural numbers.[10] $\square$

1.3.2. Lindström's generalization. Currently the term *generalized quantifiers* is used in the sense given by Per Lindström in [8]. For every finite sequence $\mathbf{t} = (t_1, \ldots, t_k)$ of positive integers, we define a generalized quantifier Q of type $\mathbf{t}$ as follows:

- On a syntactic level, for each formulae $\varphi_1, \ldots, \varphi_k$, we have a new formula $Q\mathbf{x}(\varphi_1(\mathbf{x}_1), \ldots, \varphi_k(\mathbf{x}_k))$, where $\mathbf{x} = x_1, \ldots, x_t$, $t = max(t_1, \ldots, t_k)$ and $\mathbf{x}_i = x_1, \ldots, x_{t_i}$, for $i = 1, \ldots, k$;
- On a semantic level, for each nonempty U the set Q_U contains k–tuples $(R_1, \ldots, R_k)$, where $R_i \subseteq U^{t_i}$, for $i = 1, \ldots, k$. Additionally we assume that *the logicalness condition* is satisfied, in the following sense: for each bijection $f : U \longrightarrow W$ and for each k–tuple $(R_1, \ldots, R_k)$:

$$(R_1, \ldots, R_k) \in Q_U \text{ if and only if } (f(R_1), \ldots, f(R_k)) \in Q_W.$$

For a model M with the universe U we define $M \models Q\mathbf{x}(\varphi_1(\mathbf{x}_1), \ldots, \varphi_k(\mathbf{x}_k))$ if and only if $(\varphi_1^{M,\mathbf{x}_1}, \ldots, \varphi_k^{M,\mathbf{x}_k}) \in Q_U$, where $\varphi_i^{M,\mathbf{x}_i} = \{(a_1, \ldots, a_{t_i}) \in U^{t_i} : M \models \varphi_i(a_1, \ldots, a_{t_i})\}$, for $i = 1, \ldots, k$.

Generalized quantifiers defined by Andrzej Mostowski are exactly Lindström's quantifiers of type (1).

[10] The argument given here is mine, but Per Lindström gave an argument in a similar style in a conversation with Michał Krynicki and me [9].

2. Old school and old ideas

Here I am starting an essay part of the paper, so I am changing to first person style from, usual in science, plural majesty style.

The great project of foundations of mathematics started as both philosophical and mathematical project. Some very important for the project researchers, as Bernard Bolzano and Bertrand Russell, were mainly philosophically motivated. Others, like Gottlob Frege and David Hilbert, were basing on both philosophical and mathematical traditions, which were for them not clearly separated. It is particularly striking in the case of Hilbert. He has got his hight position and influence in mathematics by purely mathematical works. However his research program of grounding foundations of mathematics by reduction to finitistic mathematics was formulated and justified in philosophical spirit. There were also other influential logicians motivated from both sides, let me mention: Rudolf Carnap, Willard Quine, Hilary Putnam and Jon Barwise.

In Poland two the most influential logicians of the first half of twentieth century, Jan Łukasiewicz i Alfred Tarski, started with philosophical problems. Jan Łukasiewicz started with the problem of justifying the basic logical laws and the problem of determinism. Alfred Tarski started with the problem of defining truth.

These were old ideas. New ideas disappeared in philosophy. Some people in Poland could think that the main reason was communistic dictature in years 1948–1989. Andrzej Mostowski openly claimed that any philosophical discussions should be removed from foundations of mathematics. The attitude can be justified by a political situation. Philosophical faculties in Poland were dominated by the communist party expositors.[11] All this unpleasant things did not touch mathematics. Stalin, who indirectly governed in Poland in 1945–1953, thought that mathematics and physics should be independent, everything else have to be penetrated by the communist party.

It is worth to mention that this attitude was not accepted by by his oldest students Andrzej Grzegorczyk and Helena Rasiowa. Andrzej Grzegorczyk went to philosophy in 1970–ties, and Helena Rasiowa strongly supported joining philosophical and mathematical interests.[12] However for all later students the attitude was obvious and acceptable.

In mathematics philosophy was replaced by bed philosophy and in philosophy philosophy was replaced by even worse philosophy.

What is important the same thing happened in many countries without communistic dictatorship. I think that the main reason is the idea of autonomy of faculties. Why we should confront relevance and importance of our results with people thinking in other way? Of course philosophy is loosing in this confrontation. However mathematics is loosing either.

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[11] I remember, when I was teenager, a comment of my father Andrzej Włodzimierz Mostowski about the book about philosophy and non–classical logics. He said: This is neither on philosophy nor on logic, this is about who should be relagated and who can keep his position.

[12] I was her student in 1982–1983.

3. The old school and new ideas — computations

The other weakness of the school of Andrzej Mostowski was not absorbing a new idea of computability. Andrzej Mostowski thought of computability not in terms of algorithms, but in a sense of arithmetically definable sets. The only his student really thinking in terms of computability was Andrzej Grzegorczyk. Reception of his ideas was very poor because he was far from the center. The center was Andrzej Mostowski. The department of Mathematical Logic guided by Helena Rasiowa reoriented to computer science in late 1970–ties.[13]

In these times the old department of foundation of mathematics practically disappeared. The department of Helena Rasiowa was quickly developing and in majority passed to the institute of computer science preserving the only strong group in logic at mathematical faculty. It is symbolic that the main results establishing position of this group in the new faculty were results related to Büchi–Rabin automata — the topic which was introduced in Poland by Andrzej Włodzimerz Mostowski.[14]

4. What for philosophy?

There is a common opinion in mathematics that a good achievement is just a difficult proof. Joining it with a view that axiomatically defined ZFC is a good basis for mathematics means that mathematics is a game for finding difficult proofs from ZF. In other words all mathematical questions are of the form: is φ provable from ZFC. More difficult proof, better result. Of course we know that ZFC is not complete. Therefore we allow questions of the form: is φ independent of ZFC.

I have asked a few mathematicians whether they would accept their activity as playing such a game. Nobody answered yes. Mathematicians lost their philosophical sensitivity. This is the only explanation of these contradictory views. They need serious philosophical thinking.

On the other hand philosophy, without real interaction with current science starts to be inferile. It is going to problems from its history. So working on history of its history, and so on.

5. Who is your master?

In late 1970–ties, when I was a young student, Krzysztof Maurin[15] asked me "who is your master?". I was surprised, I did not know what to answer. After a while

[13] Andrzej Salwicki told that they did not know works by Andrzej Grzegorczyk. Only later on they recognized his works as relevant and important.

[14] He is my father, and younger cousin of Andrzej Mostowski. Similarly as sons of Andrzej Mostowski: Tadeusz and Jan, he lived in a shadow of Andrzej Mostowski. I was the the first person in the family who took the topics of Andrzej Mostowski.

[15] He was in this time very eminent professor working in mathematical analysis and mathematical physics. He was also deeply interested in philosophy of mathematics.

I answered "Jan Łukasiewicz". Łukasiewicz has died in 1956. After a few years I would answer "Alfred Tarski", but he died either. Never in my life I met anyone of them. After some time I realized that I could not honestly answer for such a question. I would say that learned from many people, frequently older ones, but I learned also a lot from younger people.

Later on I was asked by my colleagues for giving them a problem to work on. When I was younger I answered "if you do not know it then you are not ready to work in science". Later on I have change my mind. By having a good problem and a good support you get a few years in advance in your scientific carrier. This was probably what I lost in my life.

Summarizing, we need masters, more masters than one.

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Marcin Mostowski
Department of logic
Institute of Philosophy
Jagiellonian University
e-mail: `marcin.mostowski@uj.edu.pl`