Adolf Lindenbaum, Metric Spaces and Decompositions

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Abstract. This paper revisits the life of Adolf Lindenbaum in light of new research findings, then looks at two areas among many—metric spaces, and decompositions of point sets—where his work has been underappreciated.

MSC. 01A60, 01A70, 01A73, 51-03, 54-03, 54A05

Key words. Adolf Lindenbaum, biography, metric spaces, monomorphy, decomposition, Warsaw School of Mathematics, Wacław Sierpiński

This article is dedicated to Mariusz Pandura, in gratitude for his tireless research efforts.

§1. Preface

"I think your reasoning is really interesting for its simplicity and effective character. I would just like to make one remark (which I've already mentioned to Erdős). As I recall, Adolf Lindenbaum had a more general result—a proof of the existence of $2^{\mathfrak{c}}$ sets not equivalent by countable decomposition in relation to an arbitrary family of bijective transformations—not necessarily a family of isometric transformations (and perhaps even a more general result for arbitrary cardinal numbers). I do not remember the proof at all, and also I do not remember whether the family of bijective transformations was subjected to some additional assumptions. I am under the impression (but can be completely mistaken) that Lindenbaum announced his result without proof either in an article in ***Fundamenta Mathematicae*** or in the reports of talks in the ***Annales de la Société Polonaise de Mathématique***. In any event it would be worthwhile to reconstruct and announce the result. Overall it seems to me that there is an obligation to mathematics and to the memory of Lindenbaum to encourage people to become acquainted with what Lindenbaum left behind in print and to publish proofs of results [that he] supplied without proof."

The paragraph above is from a letter Alfred Tarski wrote to Wacław Sierpiński on October 30th, 1946,[1] discussing a pre-publication copy of Sierpiński [1947], a paper which strengthened a result from Erdős [1943]: using the axiom of choice, Erdős had proved (page 644) that for every cardinal $\mathfrak{m} < \mathfrak{c}$ there is a family of $2^{\mathfrak{c}}$ sets of real numbers, no two of which can be decomposed into $\mathfrak{m}$ disjoint mutually congruent subsets. Citing something he thought

[1] More precisely, it is from our translation of Tarski's letter. The original letter was in Polish. McFarland–McFarland–Smith [2014] offer their own translation on pp. 377–379, very close to ours, but with two material differences: "efficiency" instead of "effective character", and "two" instead of "$2^{\mathfrak{c}}$". We make some changes to parts of their footnote 82 (*ibid*., page 378), while adopting other parts of it *verbatim*.

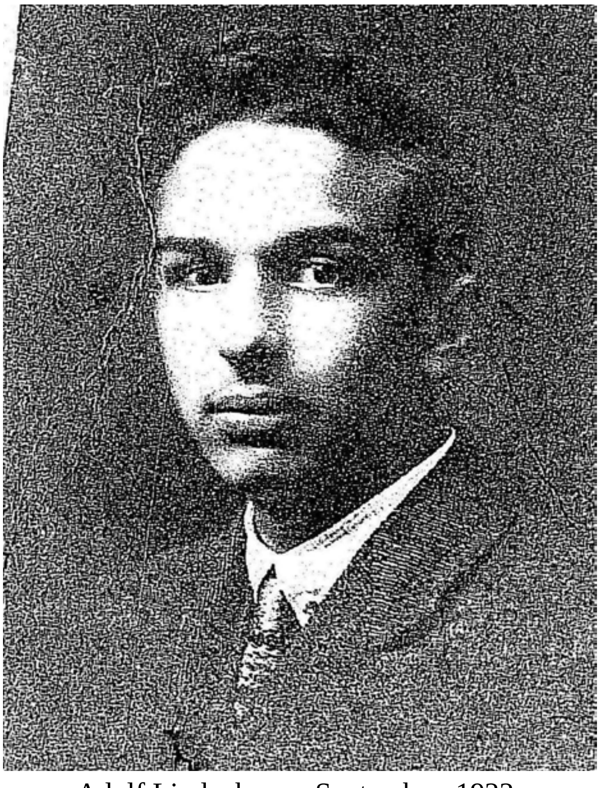
he recalled Tarski telling him, Erdős had credited Lindenbaum for announcing, without proof, a less general version of that theorem. Sierpiński [1947] obtained a more general version of Erdős's 1943 result, without using the axiom of choice. However, Erdős must have been misremembering what Tarski had told him. Tarski in fact believed that Lindenbaum had achieved an even more general result than Sierpiński's, much earlier, and without the axiom of choice.

So do we (see the concluding paragraphs of our §4 below), though we cannot find any published reference to it apart from Tarski's 1946 letter to Sierpiński. Moreover, we agree wholeheartedly with Tarski's judgment that "there is an obligation to mathematics and to the memory of Lindenbaum to encourage people to become acquainted with what Lindenbaum left behind in print and to publish proofs of results [that he] supplied without proof."

It is the object of the present article to provide just such encouragement.

§2. A Short Life

Lindenbaum, Adolf (1904–1941); Polish-Jewish mathematician and logician; *docent* of the University of Warsaw; member of the Warsaw school of mathematics and the Warsaw school of mathematical logic; early supporter of ***Fundamenta Mathematicae***; co-founder of the Polish Logical Society; Alfred Tarski's closest collaborator of the inter-war period; logical positivist and member of the Vienna Circle; member of the International Unity of Science movement; anti-war campaigner; Polish Communist Party activist; Holocaust victim.



Adolf Lindenbaum, September, 1922.

Adolf Lindenbaum, November, 1927.

Adolf Lindenbaum was born 12 June 1904 in Warsaw, the son of Mowsza Henoch aka Maurycy Henryk (1878–1932) and Emilja née Krykus (1875–?). He had a younger sister Stefanja, born 22 March 1908.[2]

Mowsza was a businessman. On Adolf's birth certificate he described himself as a "*прикащикъ*"—an accordion word that can mean shop clerk, sales assistant, steward, purser, branch manager, manager, director, superintendent, majordomo, or overseer. We surmise that Mowsza's father owned several businesses and put Mowsza in charge of one or more of them. Soon afterward Mowsza switched to describing himself as a "*kupiec*"—which simply means businessman. On a 1924 document he is named as one of the officers of the Jewish Businessmen's Mutual Assistance Society.

Adolf's sister Stefanja entered *pensja dla dziewcząt Pauliny Hewelkówny*[3] in 1917 and matriculated in 1926. She was accepted into the Faculty of Law at the University of Warsaw on 2 September 1926. She attended all three trimesters of the 1926/27 academic year, but did not sit any exams, and she formally withdrew from the university on 31 August 1927.

There is some slight evidence hinting at two more Lindenbaums in the household—an S. Lindenbaum aka Z. Lindenbaum (possibly an elder sibling of Mowsza Henoch's), born 11 December 1886; and an M. Lindenbaum, born sometime in 1912—both of whom appear to have been registered at the family's home address in the 1930s and to have emigrated to England either during or shortly after the Second World War.[4]

The family's financial circumstances were boom and bust. Mowsza was in the movie distribution and movie-theater franchising, leasing and financing businesses. From 1926 he was co-owner and general manager of *Spółka Kinematograficzna "Kolos"* (in Warsaw) and *"Kolos Małopolski"* (in Kraków). In the later 1920s Stefanja was brought into several of the businesses as a co-owner and board member, and Mowsza stepped down from some of their boards of directors in favour of his daughter. He ventured into movie production in 1931 with a production company called *Towarzystwo Kinematograficzne "Tempofilm"*, with which he had at least one box-office success that we know of.

[2] Some of §1 of this article draws on Marczewski–Mostowski [1971] and in several places is a straightforward translation of that dictionary entry—a debt which the present authors are keen to acknowledge up front. As well, some of §1 of this article overlaps with §1 of the paper Zygmunt–Purdy [2014], where readers will find a more detailed treatment of Lindenbaum's university student years, professional life and participation in congresses. That said, much of the present material is new: some of it even overturns parts of §1 of Zygmunt–Purdy [2014].

[3] A private school for girls, at that time on the corner of Marszałkowska and Sienkiewicza (ul. Marszałkowska 122). In 1919 the school was nationalized and renamed *Państwowe Gimnazjum Żeńskie im. Klementyny z Tańskich Hoffmanowej*.

[4] At the present time this cannot be substantiated, as the Polish National Archives have "masked" the relevant records, and the U.K. National Archives have "closed" them for 100 years, on the grounds that they "contain sensitive personal information which would substantially distress or endanger a living person or his or her descendants". The "slight evidence" consists of scanned pages from the registry of residents of the building at ulica Złota 45 inadvertently revealing some details that are less-than-perfectly masked, and U.K. naturalization records HO 405/33558 and HO 405/33790.

Then he ran into financial difficulties. *"Kolos Małopolski"* went into bankruptcy, with Mowsza holding half of its shares, worth at that time some 50,000 złotych. He took out some very large loans, and was unable to pay them back. He was last seen alive on 27 December 1932. His body was recovered from the Vistula River next spring when the ice melted.

"*Tempofilm*" was legally dissolved on 26 January 1934. Some of the other businesses survived. Records show that Stefanja remained a shareholder throughout most of the 1930s. But shareholders outside the Lindenbaum family were also recorded, and it is not known if Stefanja held a controlling interest, or if the businesses were profitable after Mowsza's death, or if any of his erstwhile creditors had claims on the earnings.

There is no record of the elementary school Adolf Lindenbaum attended. In his first year of secondary school, 1914–15, he attended *gimnazjum Rocha Kowalskiego*, and then from 1915 to 1922 *gimnazjum Michała Kreczmara*. In 1922 he entered the University of Warsaw, and on 22 June 1928 was awarded a Ph.D. for a thesis titled "*O własnościach metrycznych mnogości punktowych*" <On the metric properties of point sets>, written under the supervision of Wacław Sierpiński.

Lindenbaum's Ph.D. diploma reads, "*primum in mathematica, deinde in physica et in philosophia*," but in his bio-bibliography for ***Erkenntnis*** he added a telling qualification: "*Hauptfach–Mathematik*; *Nebenfächer–Philosophie, Experimentalphysik*." In other words, not simply physics, but experimental physics—involving measurement.

He took courses in descriptive geometry and projective geometry, dealing with transforms and invariants, and in mathematical astronomy and astrometry, Maxwell's equations, Planck radiation and *Lichtquanta*.[5] He knew of Hendrik Lorentz's group-theoretic treatment of frame transforms and their invariants, and the role they played in the geometrization of space-time. He wrote his thesis in the wake of the Banach-Tarski paradox, which had possibly stirred some philosophical misgivings in him about measures in physics—Hausdorff, Dirac, Liouville-Hamiltonian, Lorentz. With a nod to the Vienna Circle and verificationism he took to using the term *metrologia* (metrology) in the titles of some of his courses. Together with Edward Szpilrajn (Marczewski) he co-authored an encyclopedia entry on measurement in geometry, aimed at a general audience; at the bottom of the entry the reader is urged to consult a Polish translation of Hermann von Helmholtz's "*Zählen und Messen*", which emphasizes the empirical nature of measurement.[6] Today the term "metrology" can be encountered in the literature on the Hausdorff dimension of space-time.

Lindenbaum was a keen participant in student academic organizations and societies. Through his papers, reviews, comments and personal contacts he exerted a strong influence on younger mathematicians. Together with Alfred Tarski he became an active contributor to

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[5] The courses "*Teorja Promieniowania*" and "*Promieniowanie i Kwanty*" were taught by Czesław Białobrzeski. Mathematical astronomy and astrometry were taught by Michał Kamieński.

[6] See [35], page 595.

two schools of scientific inquiry: the Warsaw school of mathematics, under the intellectual influence of Wacław Sierpiński and Stefan Mazurkiewicz, and the Warsaw school of mathematical logic, principally under the leadership of Jan Łukasiewicz and Stanisław Leśniewski. He also kept in close contact with the Lwów school of mathematics through Stefan Banach, Hugo Steinhaus and Stanisław Ulam, and two of his publications show him encouraging and riding to the defense of a particularly promising young doctoral student of Steinhaus called Sala Weinlös.

He published mainly in ***Fundamenta Mathematicae***. His first major paper, "*Contributions à l'étude de l'espace métrique. I.*", appeared in ***FM***, volume 8 (1926), pp. 209–222. Written while he was in his fourth year and loaded down with course work, it was a remarkable accomplishment. Later he used it as part of his doctoral dissertation.

His second major paper, "*Communication sur les recherches de la théorie des ensembles,*" ***Sprawozdania z posiedzeń Towarzystwa Naukowego Warszawskiego, Wydział III***, volume 19 (1926), pp. 299–330, was co-authored with Alfred Tarski. It was a monumental work, setting out thirty pages of new results in general set theory and the arithmetic of transfinite ordinals all presented without proofs. Some were results obtained jointly by Lindenbaum and Mojżesz Dawid Kirszbraun, a classmate of his at *gimnazjum Kreczmara*. Most came from work Lindenbaum and Tarski had undertaken prior to 1926 on the theory of one-to-one transforms.

Despite the value and sheer multitude of his contributions to set theory, cardinal and ordinal arithmetic, the axiom of choice, the continuum hypothesis, theory of functions, measure theory, point set topology, geometry and real analysis, Lindenbaum's name continues to be associated principally with his work in mathematical logic, a field which in the 1920s was not yet widely developed.

His most important result in logic was his conjecture that any propositional calculus can be characterized by a denumerable (i.e., finite or at most countably infinite) matrix. He never published this conjecture. Its first appearance in print was by Łukasiewicz and Tarski in 1930. Jerzy Łoś first published a full proof of it in 1949. Lindenbaum conceived a method of constructing matrices by using the expressions of a propositional calculus (or more strictly speaking, equivalence classes of expressions) as elements of the matrix. For logicians in the 1920s, this idea was a revelation. Lindenbaum's method spawned waves of research, and it became generally accepted practice to refer to such algebras as Lindenbaum algebras.

His second most important result, widely known among logicians as "Lindenbaum's Lemma," was his theorem, framed in the terminology and concepts of Tarski's $<S,Cn>$ methodology of deductive systems, that every *Cn*-consistent set of sentences in a language *S* can be extended to form a *Cn*-consistent and *Cn*-complete deductive system in *S*. Or, more loosely put, that every consistent theory, formulated in a suitable language and assuming a suitable underlying logic, has a complete (maximal) and consistent extension. Again, as with much of Lindenbaum's legacy, he never stated it in print. Its statement and proof were first published by Tarski (see, respectively, Tarski [1928] and Tarski [1930]), who scrupulously attributed both the idea and the proof to Lindenbaum. The "Lemma" (i.e., theorem) quickly became an essential tool in every logician's toolkit. Some writers have even ventured that Lindenbaum maximalization, with the notions of completeness and consistency defined as in Tarski [1928], is the only essential thing that all logics have in common.

Lindenbaum was the co-author (with Tarski) of a 1936 paper proving that all the logical notions of Russell and Whitehead's ***Principia Mathematica*** are invariant under one-to-one transformations (automorphisms) of the domain of discourse of the model onto itself: "*Über die Beschränktheit der Ausdrucksmittel deduktiver Theorien*"—a paper which anticipated and did most of the heavy lifting for Tarski's much later work on "What are Logical Notions?" The results they obtained in this paper had many and various applications, among other things to the foundations of geometry, to discussions of which mathematical concepts count as purely logical and which as specifically mathematical, to the study of the interdependence of primitive concepts in axiomatic systems, and ultimately to the study of the independence of the axiom of choice. Lindenbaum and Andrzej Mostowski documented their findings on this last problem in 1938 in the joint paper, "*Über die Unabhängigkeit des Auswahlaxioms und einiger seiner Folgerungen*."

Lindenbaum was an adept crossover artist, equally at home pursuing the program of the axiomatists (David Hilbert, Bertrand Russell, the Italians Giuseppe Peano, Alessandro Padoa and Mario Pieri) to reduce mathematics to language, truth and logic, and the program of the algebraists (quintessentially Tarski) to reduce language, truth and logic to mathematics. The same whiff of circularity, or more charitably coherentism, can be found today in the interplay between model theory and proof theory. In his published works and public lectures Lindenbaum concentrated on large themes, fundamental issues, general concepts and synoptic solutions. He sought, throughout all, to apply whatever means necessary to achieve the clearest possible understanding of the underlying reality of things.

Lindenbaum was an adherent of Logical Empiricism. In March, 1930, he spent time in Vienna where he met Rudolf Carnap, Herbert Feigl, Carl Hempel, Abraham Fraenkel and Samuel and Lilian Broadwin. Later in the 1930s he participated in and contributed to the International Unity of Science movement. He played an active role in the movement's founding congresses in Prague (Aug 31st–Sept 1st, 1934) and Paris (Sept 15th–23rd, 1935), and he corresponded with Otto Neurath and Jørgen Jørgensen on detailed arrangements for its next two congresses, in Copenhagen (June 21st–26th, 1936), and again in Paris (July 29th–31st, 1937). Neurath twice solicited Lindenbaum's bio-bibliography, in 1930 and again in 1934, for inclusion in his surveys (published in ***Erkenntnis***) of who's who in the movement. In his article "After Six Years", ***Synthese***, vol. 5, no. 1/2 (May–June, 1946), pp.77–82 [and date-lined Oxford, December 19th, 1945, three days before his death], Neurath states outright that Adolf Lindenbaum had been a member of the original Vienna Circle.

In a letter dated July 1st, 1935, Neurath invited Lindenbaum to speak on the subject of formal simplicity (*die formale Einfachheit*) at the 1935 Paris congress. Lindenbaum obliged with a lecture of the same title, which he gave in German on the morning of September 18th in Room 1 (*Vormittag, 18 September, Saal I*). Two days later, on the afternoon of September 20th, a debate took place in the same room on the question of standardizing logical symbolism (*Aussprache über Vereinheitlichung der logischen Symbolik*). The debate concluded by agreeing to establish a working committee charged with advising on and promoting the international standardization of logical symbolism. Lindenbaum was appointed to this committee.

Records show that Lindenbaum expected to take part in the 1937 Paris congress (July 29th–31st, 1937) in his capacity as a member of this committee, but at the last moment it emerged that he was unable to attend: he was denied a travel document to leave Poland. In place of attending in person, he sent a letter which was read aloud to the congress on the morning of July 30th, expressing his and other Polish logicians' concerns with the interim results of the committee ("*meine Meinung aussprechen, wobei ich im Voraus bemerken möchte, dass die polnischen Logiker verschiedentlich anderen Standpunkt einnehmen.*").

Lindenbaum is known to have been an *asystent* in Łukasiewicz's Philosophical Seminar in the faculty of mathematics and natural sciences from (at least) the fall of 1931. Samuel Eilenberg recalled him being in charge of the library at that time.[7] In 1934 he successfully defended a habilitation thesis. Its title is lost, but there are strong indications it might have been the paper, "*Z teorii uporządkowania wielokrotnego*" <*Sur la théorie de l'ordre multiple*>, ***Wiadomości Matematyczne***, vol. 37 (1934), pp.1–35, on an extension of Cantor's notion of multiply-ordered sets. On February 1st, 1935, he started lecturing as a *docent* of the University of Warsaw, and from the commencement of the 1935–36 academic year he took up the position of *adiunkt* (assistant professor) in the Philosophical Seminar.

Lindenbaum's political sympathies were left-leaning, anti-fascist and anti-war, and some of his political activities were illegal for the time. He attended the World Congress Against War held in Amsterdam on 27th–29th August 1932. He belonged to the Polish Communist Party from at least the mid-1930s up until it was disbanded in 1938 by Stalin, and

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[7] See Eilenberg [1993], page 1. Eilenberg writes:

"In the academic year 1930–31 I was a first-year student at the University of Warsaw, while Karol Borsuk was an assistant conducting exercises in real analysis. I was a member of a class which was huge but he soon started to notice me and we got involved in several conversations. In the spring of 1931 he received his doctorate and I attended the ceremony. At the same time I attended a course on set theory given by Docent Bronisław Knaster. There were two other students in the course, however, I was the only one who did all the homework. I struck up a friendship with Knaster which lasted as long as he did. Set theory naturally led to Topology, which in Warsaw meant strictly Set-theoretical Topology.

"I remember a curious incident. In the fall of 1931 I was browsing through the Mathematics Library and I came across a book entitled ***Topology*** by Solomon Lefschetz. I looked at the bibliography to see to what extent the "Polish School" was quoted. I found only one reference. It was a paper of Knaster, Kuratowski and Mazurkiewicz in volume 15 of ***Fundamenta Matematicae*** containing a combinatorial proof of the Brouwer Fixed Point Theorem. I was very surprised to find no other references and I conveyed my concern to Dr. Adolf Lindenbaum (an excellent logician) who was then the assistant in charge of the library. He told me that it was a terminological misunderstanding, that the book was not about Topology but about some sort of algebra."

campaigned for it in intelligentsia circles. In 1936, as one of the “editors and co-workers” of ***Głos Współczesny***, he signed a petition to Professor Halvdan Koht of the Nobel Committee in Oslo, urging that Karl von Ossietzky, a German political journalist imprisoned by the Nazis, be awarded the Nobel Peace Prize.[8] Together with many writers and social activists of the day Lindenbaum added his name to an open letter “to the workers of Lwów” expressing solidarity with “the proletariat’s protest against the bloody massacre [of April, 1936] of workers fighting for jobs, bread and freedom”.[9]

Lindenbaum took a keen interest in pedagogy, and in the second half of the 1930s he delivered various lecture series to teachers’ groups and organizations. He was also said to have been interested in art, literature, hiking and mountain climbing. One of his lecture series to the Polish Teachers Union included a lecture on the creative and “artistic” elements in mathematics. It seems he enjoyed teaching beyond the ranks of his own profession, and especially teaching other teachers. It appears, too, that he was an avid solver of newspaper chess problems: his name pops up time and again in the mid-1920s in ***ABC Nowiny Codzienne*** and ***Nasz Przegląd*** for having solved the previous issue’s challenge puzzle. He even published a mathematical paper, “*Sur le «problème fondamental» du jeu d’échecs*,” ***Rocznik Polskiego Towarzystwa Matematycznego*** (= ***Annales de la Société Polonaise de Mathématique***), vol. 13 (année 1934, publ. 1935), pp.124–125. At the same time it was rumored he liked “having a good time”, being among people, frequenting cafés and partying. Interestingly though, in group photos he was never front row center. He seemed to prefer the third or fourth row, or somewhere over by a wall.

Around the end of October or the beginning of November 1935 Lindenbaum married Janina Hosiasson, an established philosopher of logic, fellow member of the Lwów-Warsaw school of mathematical logic, fellow graduate of the University of Warsaw, and four-and-a-half years older than him. She also was named by Neurath as having been a member of the original Vienna Circle. During their engagement they stayed for two or three months with Adolf’s mother and sister in the Lindenbaums’ family home at Złota 45/4 before moving into their own apartment at Krasińskiego 16/34 on the 31st of October, 1935.

It was, according to Antoni Marianowicz (Kazimierz Jerzy Berman), a marriage of convenience for both of them.[10] Janina was in love with Antoni Ludwik Pański (1895–1942), philosopher, statistician, social activist and eldest son of the neurologist Aleksander Pański. Janina and Antoni had been living together, on again off again, since Janina’s late teens. During this time Antoni had a string of affairs with other women, and was briefly married (to

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[8] The petition was issued in the name of the editorial staff, co-workers and “friends” of ***Głos Współczesny*** <*Contemporary Voice*>, a left-leaning monthly newspaper with national circulation, and was splashed across the front page of the March 1936 issue. The signatures of editorial staff and co-workers were grouped separately from those of “friends”. Lindenbaum’s signature was included among editorial staff and co-workers.

[9] Published in ***Lewar***, 15th May, 1936, no. 4, page 10. ***Lewar*** was a biweekly literary magazine sponsored and influenced by the Polish Communist Party from 1933 through 1936. Its name was a play on words, combining “leverage” and “leftist”.

[10] See Marianowicz [1995], pp. 230–231, or Marianowicz [1995] (1), pp. 192–193.

a singer, Elza Aftergut), but his love for Janina was apparently "the real thing". Awkwardly, they were second cousins.[11] Marianowicz writes that their respective "aunts and uncles" pressured them to choose other marriage partners.

Marianowicz does not spell out why he thinks it was a marriage of convenience for Adolf. It is not clear if Adolf regarded the marriage the same way Janina did. To all outward appearances they cohabited just fine, and seemed to care for one another. Certainly they held each other's professional abilities in high regard. They were fired by the same progressive social ideals and political convictions. But was Janina the apple of Adolf's eye?…the "little man" (אישון עין)[12] of his eye? We are indebted to Arie Hinkis for his suggestion that Adolf was delicate, feminine, the exact opposite of Tarski; that he exhibited none of Tarski's machismo, competitiveness or ego; and that only Tarski could have "extracted" from the young Adolf the prodigious achievements of [26] and [26a]. Perhaps Lindenbaum was under social pressures of a different kind.

On September 6th, 1939, Adolf and Janina abandoned their apartment and all their belongings and fled Warsaw on foot,[13] heading (Janina's letters to Otto Neurath and G.E. Moore suggest) either due east in the direction of Siedlce, or south-east in the direction of Dęblin. Janina writes to Neurath and Moore that progress on foot was slow and that the road was repeatedly strafed by Luftwaffe planes. A friend with a motorcycle encountered them on the road and gave Janina a ride on the saddle behind him, leaving Adolf to continue on foot. The motorcyclist managed to get Janina as far east as Rivne[14] where he left her before heading back alone in the direction of Warsaw. From Rivne, Janina made her way, catch as catch can, partly by train and partly by road, to Vilnius. There she eventually learned through friends and acquaintances that Adolf was in Białystok.[15] Adolf and Janina tried writing to each other, without much success. Apparently, most of their letters to each other were not allowed to get through—confiscated by one postal authority or another. Janina visited Adolf in Białystok for one day, then she returned to Vilnius without him. Her letters to Neurath and

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[11] Janina's mother Zofja Hosiasson, née Feigenblatt, and Antoni's mother Róża Pańska, née Seidemann, were first cousins—(Antoni's maternal grandfather Adolf Seidemann and Janina's maternal grandmother Leona Feigenblatt, née Seidemann, were brother and sister).

[12] The biblical Hebrew origin of today's expression (see: Deuteronomy 32:10, Psalms 17:8, Proverbs 7:2, Lamentations 2:18).

[13] German forces invaded Poland on September 1st and within days German artillery shells were raining down on Warsaw. Janina and Adolf fled their home under fire.

[14] Polish: Równe; Ukrainian: Рівне; Russian: Ровно; Hebrew: רובנו; Yiddish: ראוונא—a name that reverberates in Aliyah consciousness. In 1939 it was a small city of about 90,000 people, roughly half of whom were Jewish.

[15] On September 17th Soviet forces entered Poland and on September 22nd Białystok came under Soviet occupation. We do not know if Lindenbaum was already in Białystok by then, or if he arrived after it was in Soviet hands. It would have been a difference of only a few days. In either case it would have suited him, as he was attracted by Soviet communism—or at least, by his imagined picture of it.

Moore suggested that she and Adolf had “agreed to disagree” about where best to try to survive.[16]

Adolf found work in the Pedagogical Institute in Białystok as a mathematics teacher and the head of its mathematics department.[17] Then on June 22nd, 1941, Germany invaded the Soviet Union, and on the same day, the Vilnius Uprising began.[18] Within days, German forces were in Białystok, and not much later in Vilnius. Sometime before the beginning of July, 1941, Adolf came to Vilnius and stayed, possibly for about six weeks, in a small satellite community on the eastern outskirts of the city called Pavilnys (Polish: Kolonia Wileńska).[19] By coincidence Pavilnys was where Anna Borkowska, aka Mother Bertranda, famously hid members of HaShomer HaTza'ir in her Dominican convent. However, there is no indication that Adolf had any knowledge of this.

Why he came when he did, and indeed at all, and why he chose to stay in Pavilnys, rather than with his wife in her apartment downtown,[20] remains unclear. Perhaps he was finally persuaded of the wisdom of trying to emigrate to the West, and hoped that Janina’s contacts in Vilnius might help him do so, but was wary of putting her in danger by openly associating with her. Sara Bender writes that under the Soviets there was a systematic campaign of destruction and arrests in Białystok from May to June, 1941, cut short only by the German invasion, and that a “fourth wave of arrests began on the night of June 20th, 1941, when members of the NKVD went from house to house with their lists, sending entire families, most of them Jewish, in cattle and freight trucks to the Soviet hinterland.” Then on June 22nd, 1941, “the bombing of Białystok sowed panic in the city. As the Red Army began to flee, […] anyone who could, fled with the Russians”.[21]

Possibly he summoned his sister Stefanja to join him in Vilnius from wherever she had been hiding. Or possibly she was already staying with him in Białystok and simply came with him. We don’t really know where she had been staying before this, or how or why she came to Vilnius when she did. All we know is that they both showed up together. Sometime before the middle of August, 1941, Adolf and Stefanja were arrested and shot. The timing of

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[16] On September 19th,1939, Soviet forces wrested the city of Wilno from Poland and on October 28th re-attached it to its ancestral home of Lithuania, upon which they bestowed (notional) “independent statehood”, a dubious arrangement which lasted only until August 3rd, 1940.

[17] The Pedagogical Institute was a Soviet institution.

[18] In Zygmunt–Purdy [2014], §1, p. 299, we wrote that on June 22nd, 1941, Germany “declared war” on the Soviet Union. Germany of course did no such thing. It merely invaded, without bothering to observe such niceties as telling anyone what it was doing. We thank Piotr Wojtylak for pointing this out.

[19] We have this on the authority of Professor Bogusław Wolniewicz, who cites testimony of Professor Maria Renata Mayenowa (born Rachela Gurewicz), from a conversation he held with her on 26 April 1986. See Wolniewicz [2015].

[20] Remarks attributed to Oskar Lange suggest that Janina and Antoni Pański were living in the same apartment in Vilnius at that time, which might explain why Adolf chose to live elsewhere.

[21] See Bender [1997] (1), pp. 87–90.

their arrests and murders—Adolf and Stefanja together—suggests that Stefanja was staying with Adolf in Pavilnys.[22]

There is some evidence to suggest that Janina finally did marry Antoni Pański… either immediately upon learning of Adolf's death, or sometime earlier. In September, 1941, both Antoni and Janina were arrested—he first, she a week later. She had two passports in her possession at the time of her arrest, one in the name of Janina Lindenbaumowa, the other in the name of Janina Pańska. Possibly Adolf had given her a divorce. Or she had committed bigamy. Or one or both of the passports, or supporting documents used to obtain them, had been forged.

Jerzy Dadaczyński writes that Janina applied for an American visa.[23] We know from American sources[24, 25] that she repeatedly tried, to no avail, to be sponsored into the U.S. by the Rockefeller Foundation's New School for Social Research as a "refugee scholar". Rudolf Carnap, Oskar Lange, William Gruen, Ernest Nagel, Albert Hofstadter, Alfred Tarski, Henry S. Leonard, Herbert Feigl, Mason W. Gross, G.E. Moore, C.A. Baylis, Carl Hempel, J.C.C. McKinsey, Sidney Hook, Willard V.O. Quine, Victor F. Lenzen and C.J. DuCasse all wrote letters to the Rockefeller Foundation supporting her applications.[26] In the end, the New School decided it was willing to accept her application to enter the U.S., but could not itself provide financial support for her, or recommend that the Rockefeller Foundation provide financial support. Dadaczyński also writes that "probably" her efforts to help her husband attracted the attention of the Lithuanian authorities that led to her arrest. What those efforts were, we do not know.

Antoni Pański died in Lukiškės prison[27] in Vilnius on January 9th, 1942 while under interrogation, probably tortured to death. His death certificate, issued by the prison hospital staff, and written in Lithuanian handwriting,[28] identified the cause of death as "*Širdies raumens degeneracija*" <heart muscle degeneration>, which probably just meant his heart stopped beating.

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[22] According to the *Dédicace*, ***Fundamenta Mathematicae***, vol. 33 (1945), p.v, Adolf Lindenbaum was shot in Naujoji Vilnia (Polish: Nowa Wilejka). This was where Soviet forces withdrew to on October 28th, 1939, after handing Vilnius over to a notionally independent Lithuania. Naujoji Vilnia is one train stop—4 kilometers—east of Pavilnys/Kolonia Wileńska, and larger. However it must be underlined that by August, 1941, German forces were in control of both places. Stefanja and Adolf were not shot by the Soviets. They were shot either by Germans or by Lithuanian collaborators.

[23] Dadaczyński [2003].

[24] SUNY, University at Albany, Science Library 352, M.E. Grenander Department of Special Collections and Archives, German and Jewish Intellectual Émigré Collection (GER-017), Series 4: individual files from Else Staudinger, Director of the American Council for Émigrés in the Professions (ACEP), Box 3, folder 147.

[25] The Rockefeller Archive Center, the Rockefeller Foundation (RF) Archives collection, Record Group 2, RG2 1940, Series 200, Box 192, folders 1368 & 1369; and RG2 1941, Series 200, Box 212, folder 1487.

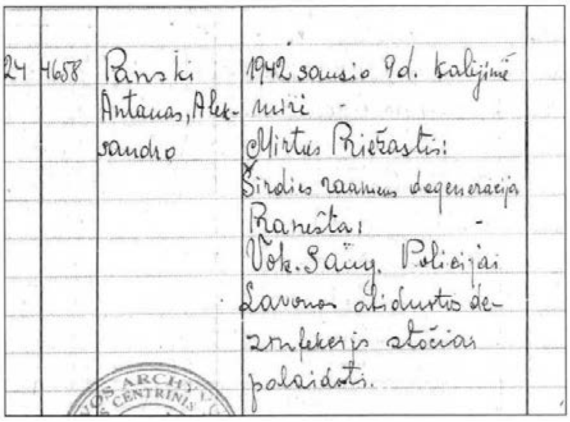
[26] Lord Russell, three of whose books Janina had translated, and himself an "economic immigrant" in the U.S. on a work visa, declined to support her, claiming that he did not recall her.

[27] Lithuanian: *Lukiškių tardymo izoliatorius kalėjimas*. Polish: *więzienie na Łukiszkach*, or simply *Łukiszki*. It was on the same street—Gedimino—as Janina's apartment.

[28] The prison and its hospital were under German direction and control, but staffed by Lithuanians.

Janina's friends engineered a prison break for her, but she fluffed it. In April, 1942, after seven months of imprisonment, she was transported to Paneriai (Polish: Ponary) and shot.

Journal entry recording the death of prisoner #4658, Antoni Pański on 9 January 1942 in the prison hospital. Monika Tomkiewicz, ***Zbrodnia w Ponarach 1941–1944***, **Monografie Komisji Ścigania Zbrodni Przeciwko Narodowi Polskiemu**, vol. 43. Instytut Pamięci Narodowej: Warszawa 2008. Fig. #24 of 51 figures on 17 unnumbered pages at the end of the book. Reproduced by kind permission of the author and Instytut Pamięci Narodowej.



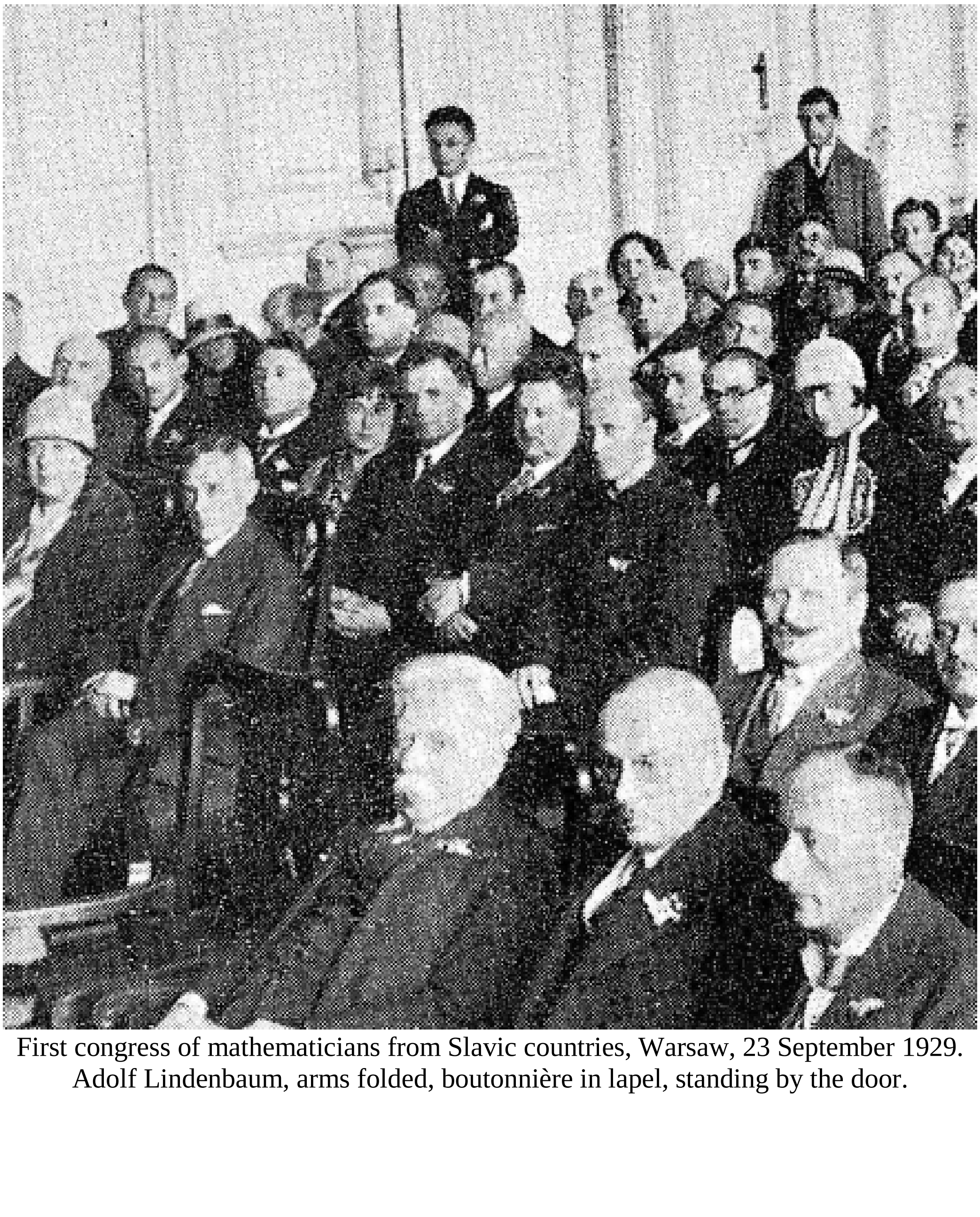
24	4658	Panski Antanas, Aleksandro	1942 sausio 9d. kalėjime mirė - Mirties Priežastis: Širdies raumens degeneracija Ranešta: - Vok. Saug. Policijai Lavonas atiduotas dezinfekcijos stočiai palaidoti.	

24. Wpis do rejestru zgonu nr 4658 więźnia Antoniego Pańskiego zmarłego 9 stycznia 1942 r. w szpitalu więziennym

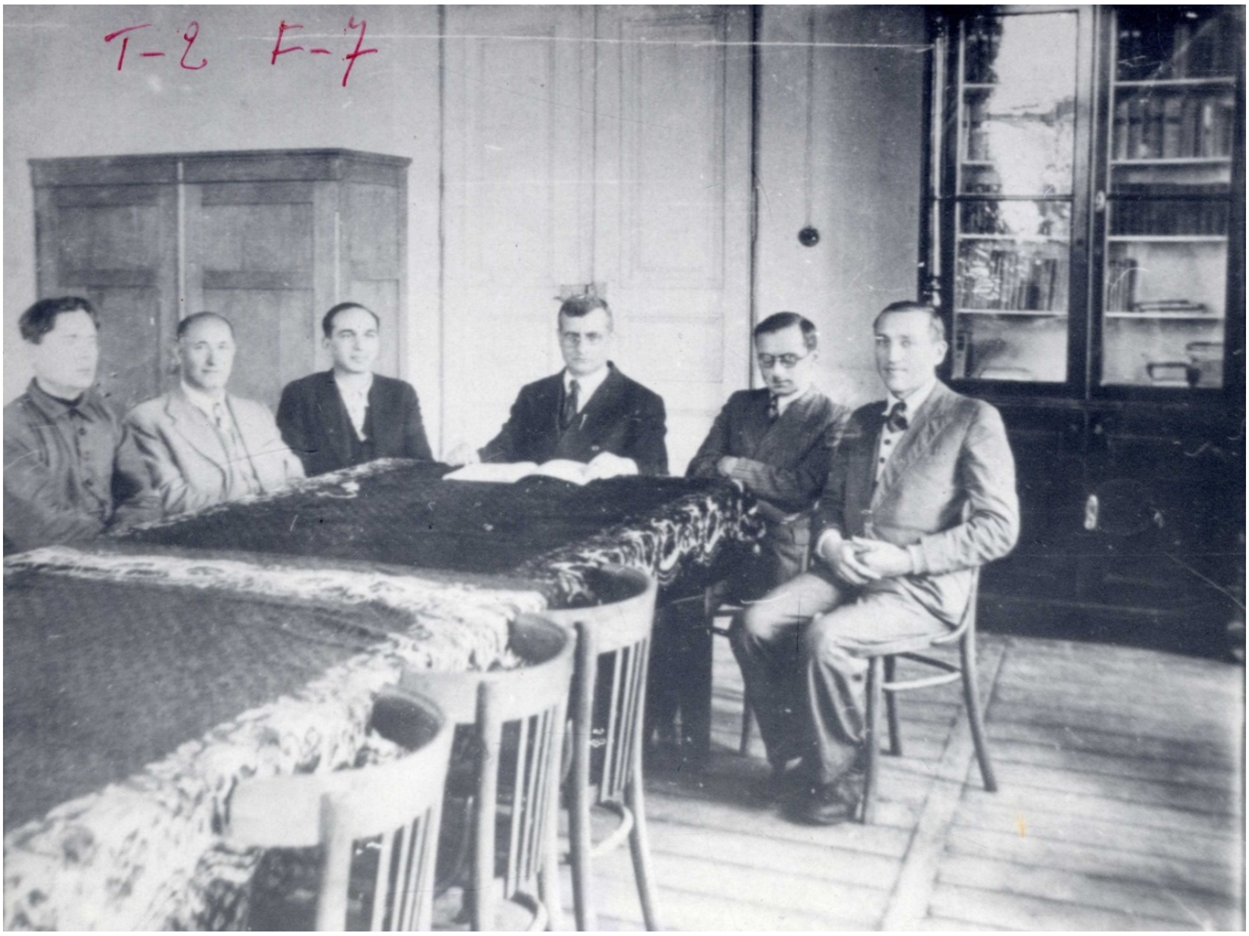
Janina Hosiasson, photograph from her university student "*indeks*", October, 1919

Stefanja Lindenbaum, photograph from her university student "*indeks*", October, 1926


The three Pański brothers, from left to right: Jerzy, Antoni, Wacław (Solski), *circa* 1920


First congress of mathematicians from Slavic countries, Warsaw, 23 September 1929.
Adolf Lindenbaum, arms folded, boutonnière in lapel, standing by the door.


Meeting of the mathematics, physics and astronomy circles in Warsaw, 5 May 1932.
Łukasiewicz, Leśniewski and Tarski seated, front row.
Lindenbaum standing two rows back, to the right, behind two gloved men in wool coats.



Pedagogical Institute, Białystok, spring of 1941. Left to right: Stanisław Romanowski, Samuel Steckel, Adolf Lindenbaum, Salomon (Szlama) Lubelski, Henryk Ferencowicz, Edward Litwinowicz.

Здание Белостокского педагогического института. 1940 г.

Pedagogical Institute in Białystok, 1940.

§3. Metric Spaces[29]

Metric spaces interested Lindenbaum from early on, as can be seen from:

- his first published paper, [26], "*Contributions à l'étude de l'espace métrique, I*",[30] later incorporated into his doctoral dissertation
- §5: "*Théorie des ensembles des points*" of the paper [26a], "*Communication sur les recherches de la théorie des ensembles*", co-authored with Tarski. §5 sets out results pertaining to the decomposability of sets and their congruence in metric spaces, some of which results were obtained by Lindenbaum alone, some jointly with Tarski, and some jointly with M.D. Kirszbraun
- his doctoral dissertation, "*O własnościach metrycznych mnogości punktowych*" <On metric properties of point sets>, submitted in 1927 and defended in 1928 (though never published in its entirety)
- the short note [29^{a}a], lifted from his doctoral dissertation, summarizing his talk at the First Polish Mathematical Congress in Lwów in 1927
- the short note [31^{a}a], "*La projection comme transformation continue la plus générale*"
- the short note [31^{a}b], "*Sur les figures convexes*"
- the lecture [31^{1}a], "*Badania nad własnościami metrycznymi mnogości punktowych*", given at the Second Polish Mathematical Congress in Vilnius (Wilno) in 1931
- the paper [33a], "*Sur les ensembles localement dénombrables dans l'espace métrique*"

It seems certain Lindenbaum's interest in metric spaces was sparked most of all by Tarski[31] and also considerably by Banach and Kuratowski.[32] Lindenbaum had a ringside seat at the gestation and birth of Tarski's "*O równoważności wielokątów*" <On the equivalence of

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[29] Metric spaces and decompositions are only two of many areas where Lindenbaum's work is underappreciated. The exigencies of the present publication limit our treatment to these two areas. Future articles will consider his contributions to sentential logics, metalogic, the general theory of sets, and the independence of the axiom of choice.

[30] The Roman "I" in the title implies that a sequel was planned. In footnote 1, p.214 (see also footnote 1, p.218) Lindenbaum indicated what he intended the sequel to be about: it was going to be a study of the notion of the equivalence of point sets by decomposition, in the sense of Banach and Tarski (see also [26a], p.327). In the event, no sequel was ever published. Nor was his doctoral dissertation ever published in its entirety, although both he and Sierpiński expected it to be: they each referred to it as "à paraître" (see [33a], p.106, note 18; and Sierpiński [1936a], p.32). As there is no surviving copy in Warsaw University's archives, or anywhere else we know of, we are unsure precisely how his dissertation related to his publications. Many authors, including Lindenbaum himself, credited his dissertation for results that were never set out in any of his (other) publications (see, e.g., Aronszajn [1932], p.99, note 12″; Kirszbraun [1934], p.78, note 4, and p.102; Sierpiński [1936a], p.32; Lindenbaum [33a], p.106, note 18).

[31] Lindenbaum and Tarski's collaboration on decomposability and congruence of point sets in Euclidean and general metric spaces began as early as 1923, when Lindenbaum was a freshman/sophomore (see [26a], p.327).

[32] In [26], p.210, one reads, "Je termine cette préface par remercier MM. Kuratowski et Tarski, qui ont bien voulu prendre intérêt à ces recherches: j'en ai profité beaucoup." Recognition of Kuratowski's impact on [26] is also expressed on p.216 in a parenthetical suffix to the statement of Théorème 7 ("C'est M. Kuratowski qui a su généraliser de cette manière intéressante une idée de ma démonstration primitive du th.8") and on p.222, note 1 ("Le problème auquel le théorème (II) donne réponse m'a été posé par M. Kuratowski.")

polygons> and Banach and Tarski's famous so-called paradox, "*Sur la décomposition des ensembles de points en parties respectivement congruentes*", the latter of which explicitly mentioned a result obtained by Lindenbaum and thus constituted, in a "proxy" sort of way, Lindenbaum's first published result. (We cite this reference below.)

On the other hand Wacław Sierpiński, Lindenbaum's PhD thesis supervisor, probably did not exert a formative influence on Lindenbaum's interest in metric spaces… at least not at first. Only when as editor of ***Fundamenta Mathematicae*** he recognized good work in [26] and agreed to guide Lindenbaum's doctoral dissertation did Sierpiński begin to exert an influence on the direction of his pupil's inquiries. And in due course vice-versa: by [33a] and Sierpiński [1933], their mutual influence on each other started to show.

In the mid-1920s when Lindenbaum began investigating metric spaces the theory was still in its formative stages of development. Maurice Fréchet introduced the concept, using different terminology, in his doctoral dissertation in 1906. Felix Hausdorff, in his ***Grundzüge der Mengenlehre***,[33] was the first to use the expression *metrischer Raum* in place of Fréchet's terminology to denote such entities.[34] Lindenbaum's [26] can thus be seen as one of its earlier systematic treatments.

We present a selection of Lindenbaum's results from [26] and [33a]. To do this, we introduce some necessary terminology.

A space M is a collection of undefined entities called points, and a *metric space* in the sense of Fréchet and Hausdorff is a pair $<M, \rho>$ consisting of a space M and a real-valued non-negative function ρ on the cross product $M \times M$ satisfying the following conditions:

(M1) $\rho(x, y) = 0$ if and only if $x = y$ (law of coincidence)

(M2) $\rho(x, y) = \rho(y, x)$ (law of symmetry)

(M3) $\rho(x, y) + \rho(y, z) \geq \rho(x, z)$ (the triangle law)

The function ρ is called a *distance function*, and the number $\rho(x, y)$ is called the *distance* between points x and y. The triangle law (M3) is so called because of its formal expression of Euclid's Proposition I. 20: "Any two sides of a triangle are together greater than the third side." The three laws as formulated above are consistent and independent.

Lindenbaum observed (see [26], p. 211) that both the law of symmetry (M2) and the stipulation that ρ be non-negative can be derived, and thereby dispensed with, by formulating the triangle law *d'une façon plus avantageuse*. Thus he showed that two independent axioms sufficed:[35]

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[33] See Hausdorff [1914].

[34] The terms themselves—*l'espace métrique* and *metrischer Raum*—were not new. *L'espace métrique* even predated Fréchet's dissertation. See Couturat [1905], page 204; and Couturat [1905] (1), page 216. However, these earlier usages were not related to the theory of metric spaces discussed here.

[35] In the 1920s Stanisław Leśniewski and Jan Łukasiewicz, among others, advocated minimizing the number of axioms in formalized deductive systems. For more on this issue see, e.g., Sobociński [1955].

(ML1) $\rho(x, y) = 0$ if and only if $x = y$ (law of coincidence)
(ML3) $\rho(x, y) + \rho(x, z) \geq \rho(y, z)$ (modified triangle law).[36]

More precisely, he showed that: *If* ρ *is a real-valued function defined on* $M \times M$ *and satisfying* (ML1) *and* (ML3), *then the laws* (M2) *and* (M3) *hold, and* ρ *takes only non-negative values, which is to say,* <*M*, ρ> *is a metric space in the usual sense.*

A metric space <*N*, σ> is said to be a *subspace* of a metric space <*M*, ρ> iff

$N \subseteq M$, and
σ is a restriction of ρ to $N \times N$.

It is obvious…

that every subset of *M* determines a *unique* restriction of ρ,
that any such restriction satisfies (ML1) and (ML3), and hence
that every subset of *M* determines a *unique* subspace of <*M*, ρ>.

Consequently it is common practice to speak of a subset $N \subseteq M$ as being a *subspace* of *M* (and *M* a *superspace* of *N*) without explicitly presenting them as ordered pairs <*M*, ρ> and <*N*, σ>, or mentioning their distance functions. Moreover, when their distance functions are explicitly mentioned, they are often presented using the same symbol to designate both the distance function on the superspace *M* and its restriction to the subspace *N*, provided this leads to no confusion.[37]

The relation of being a subspace is transitive. Hence, in regarding a subset *N* as a subspace of *M*, we need not consider whether the metric on *N* is inherited directly from *M*, or indirectly, from the metric on some intermediate subspace *Q*, where $N \subseteq Q \subseteq M$. Thus, if *N* is a subset of *M*, we may refer to *N* as either a subset or a subspace of *M*. The choice is one of emphasis only. If we refer to the *subspace N*, we are focusing our attention primarily on *N* itself, *qua* metric space, whereas if we refer to the *subset N*, we are considering *N*'s set-theoretic properties in relation to *M*.

[26] began by defining basic notions of point-set topology and metric spaces. It cited Hausdorff's 1914 classic ***Mengenlehre***, but in its choice of definitions, which Lindenbaum conceded "*diffèrent souvent de celles qu'on trouve ailleurs*" <often differ from those found

[36] He noted two other *avantageuses* modifications which could as well do the trick—"$\rho(x, z) + \rho(y, z) \geq \rho(x, y)$" and "$\rho(z, x) + \rho(y, z) \geq \rho(x, y)$"—the second of which he credited to Piotr Szymański. Garrett Birkhoff, citing Lindenbaum's [26], gave yet another modification (see Birkhoff [1944], p.466): "$\rho(x, y) + \rho(y, z) \geq \rho(z, x)$". Birkhoff thought this "circularity postulate," as he called it, had "a clear intuitive content: if one journeys from *p* to *q* and then from *q* to *r*, the minimum energy required to get back to *p* is not more than that already expended."

[37] A similar shorthand is frequently adopted in speaking of metric spaces of differing dimensions, where the space of lower dimension can be considered as embedded in the higher-dimensional space: in this case, too, their distance functions are often presented using the same symbol to designate both.

elsewhere>, it was mainly motivated by a desire to avoid using the axiom of choice (AC).[38] This is particularly evident in its definitions of a closure operation and a compact set.

Namely, if A is a subset of M, then the *closure* $\bar{A}$ of A is the set $A \cup A'$, where A' is the derived set of A in the space $<M, \rho>$, i.e., the set of all accumulation points of A in $<M, \rho>$.[39] Hence the closure operation is not defined in the spirit of Fréchet as the set of all limit points of A (or points "adherent to" A).[40]

Lindenbaum defined compactness by means of the *Cantor condition*: a metric space $<M, \rho>$ is said to be *compact* iff, for every finite decreasing sequence $\{F_k\}$ of non-empty closed subsets of M, $\ldots F_{k+1} \subseteq F_k \subseteq \ldots \subseteq F_1 \subseteq M$, the intersection $\bigcap\{F_k\colon k < \infty\}$ is non-empty.[41] And he commented that an alternative definition might read as follows: a metric space $<M, \rho>$ is *compact* iff either M is finite or, for every infinite subset X of M, the derived set X' is non-empty (i.e., every divergent subset of X is finite).[42]

An *isometric transformation* or *isometry* between metric spaces $<M, \rho>$ and $<N, \sigma>$ is a surjective mapping f between the points of M and the points of N which preserves distance. Thus:

(*) $\sigma(f(x), f(y)) = \rho(x, y)$ for every pair x, y of points in M.

It is easy to see that any f satisfying the above conditions must be one-to-one. So, for any function $f: M \to N$, if f satisfies (*) then it is an isometry between $<M, \rho>$ and the subspace $<f(M), \sigma>$ of the space $<N, \sigma>$… called the *image space of M in N under f*. One also says that such an f is an *isometry of M into N*.

Metric spaces $<M, \rho>$ and $<N, \sigma>$ are said to be *isomorphic*, or *congruent*, or *superposable* (in symbols $<M, \rho> \cong <N, \sigma>$; or simply $M \cong N$), iff there exists an isometry between them.

Notice that any two subspaces A and B of the real line with the standard or "natural" distance function $\sigma(x, y) = |x - y|$ are superposable only by means of a translation or rotation.

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[38] This attitude toward the axiom of choice is stated explicitly in [26] on page 212, footnotes 1 and 3.

[39] Let $a \in M$ and $A \subseteq M$. Recall that a is an *accumulation point* of A in the metric space $<M, \rho>$ iff every open sphere with centre a contains at least one point of A which is distinct from a (and consequently an infinite number of points of A). It is easy to see that a's being (or not being) an accumulation point of A does not depend on the whole space M, but only on the subspace $A \cup \{a\}$.

[40] As a student of Sierpiński, Lindenbaum must certainly have known that the theorem "If A is closed ($=\bar{A}$), then A contains all its limit points" is provable without using the axiom of choice, whereas the proof of the converse implication needs AC. See Sierpiński [1918].

[41] To be more precise, Lindenbaum defined what it means for an arbitrary subset $A \subseteq M$ to be *compact in a metric space* $<M, \rho>$. Then if A is *closed and compact in* $<M, \rho>$, then the *subspace* $<A, \rho>$ is compact. In general the assumption of closedness cannot be omitted.

[42] One can prove without AC that if a metric space is compact by the first definition, using the Cantor condition, then it is also compact by the second definition. The proof of the converse implication requires AC. In general topology, a topological Hausdorff space (a T_2 space) satisfying the Cantor condition, or equivalently the dual Borel condition for open sets, is called *countably compact*. In the class of metric spaces, compactness and countable compactness are equivalent.

That is, if <*A*, ρ> ≅ <*B*, ρ> then the isometric transformation *f* establishing this congruence is either $f(x) = x + c$, or $f(x) = -x + c$, where *c* is a constant.[43]

An interesting theorem due to Sierpiński states: *Any linear set A contains no more than one point p such that* $A - \{p\} \cong A$. A corollary states: *In any non-empty linear set A there exists a point p such that A is* <u>*not*</u> *congruent to* $A - \{p\}$. See Sierpiński [1954], page 7.

The congruence relation ≅ is of course nothing more than the familiar concept of isomorphism as applied to the class of metric spaces. It is an equivalence relation on this class. But the class of metric spaces admits of another, more inclusive[44] equivalence relation, namely *homeomorphism*, which turns out to be much more important than simple isometry.

Aware of this, Lindenbaum devoted the second half of §3 of [26] to "the topological properties of congruence and the problem of extending a given congruence" (see p. 214), by which he apparently meant, laying some groundwork for relating congruence to topology. Then in §4 he turned to the concept of *monomorphism*. We employ our own numbering for the results of these last nine pages of [26], departing from Lindenbaum's original, slightly shambolic scheme:

Th 1. *Every isometry is a homeomorphism*.

Th 2. *If an isometry f maps a compact set A onto a compact set B, then there exists a unique isometry f* of* $\bar{A}$ *onto* $\bar{B}$ *which is identical with f on A.*

Hausdorff [1914] defined a *totally bounded* metric space as: <*M*, ρ> is called *totally bounded* iff, for every ε > 0, there is a finite subset $A \subseteq M$ such that for every $x \in M$ there is an $a \in A$ with $\rho(x, a) < \varepsilon$. This is equivalent to the condition that, for every ε > 0, *M* is the union of a finite number of open spheres (or balls) of radii < ε. Any compact metric space is totally bounded but not all totally bounded metric spaces are compact.

Th 3. *If* <*M*, ρ> *is a totally bounded metric space, and* $f: M \to M$ *is an isometry of M into itself, then the image f* (*M*) *is dense in M,* i.e. $\overline{f(M)} = M$.[45]

To prove Th 3, Lindenbaum used the Dedekind chain method, which was used extensively in the 1920s by Sierpiński, Kuratowski and Banach.

In §4 of [26], a set is said to be *monomorphic* iff it is not congruent with any of its proper subsets. Lindenbaum defined it thus: "The set *A* is *monomorphic,* if the relations $B \cong A$ and $B \subseteq A$ occur only when $B = A$" (Def. 3, p.217), and he added that a set is monomorphic if it is a minimal ("*irréductible*") element in the class of all sets on which it is superposable.

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[43] …though there are plenty of non-standard distance functions for which this is not true (readers of a certain age may recall slide rules). Subspaces of the real line with the standard distance function are called *linear* sets (not to be confused with *linearly ordered* sets).

[44] An equivalence relation is said to be more (/less) inclusive iff the corresponding partition has coarser (/finer) granularity.

[45] This readable formulation of Lindenbaum's Théorème 7 (p.216) is due to Ryszard Engelking [1989], p.278.

It follows that monomorphy (or non-monomorphy), although being a property of a set *qua* metric space, does not depend on the distance function used to construct a metric space out of the set; it inheres in the nature of the set itself, not in the nature of the distance function. Hence the definition should more correctly be worded thus: a metric space is monomorphic iff it is not congruent with any of its proper subspaces. Alternatively: iff all distance-preserving transformations of the space into itself are surjective.

Lindenbaum remarked that the following hold for Euclidean spaces:

(a) There exist non-monomorphic linear sets;
(b) Every bounded linear set is monomorphic;
(c) There exists a bounded plane set which is not monomorphic.

He then derived some sufficient conditions for a metric space to be monomorphic.[46] As mentioned earlier, we use a different numbering scheme from his:

Th 4. *If <M, ρ> is a compact space, then it is monomorphic.* Alternatively: *Any closed and compact subset of a metric space is monomorphic.* (Théorème 8 in his original numbering scheme)[47]

Th 5a. *Every bounded subset of a compact metric space which is both F_σ and G_δ is monomorphic.*[48]

Th 5b. *Every bounded set in an n-dimensional Euclidean*[49] *space which is both F_σ and G_δ is monomorphic.*[50]

Several theorems in §5 of [26a] are closely related to [26]: We mention three of them here: theorem **4,** due to Kirszbraun and Lindenbaum, and theorems **5**(*L*) and **14**(*L*), due to Lindenbaum alone.[51] The first two each give sufficient conditions for an *expanding*[52] mapping to be an isometry on a Euclidean space:

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[46] In his words, "*Le théorème 8, les corollaires 15 et 14, nous fourniront des conditions suffisantes, de plus en plus générales, pour qu'un ensemble compact soit monomorphe*."

[47] Th 4 was generalized by Tarski as theorem **17**(*T*) in [26a], p.329.

[48] In the Introduction to [26] Lindenbaum wrote, "*Au* §4 *j'examine la propriété singulière d'un ensemble de points d'<u>être superposable avec son vrai sous-ensemble</u>. On peut indiquer des ensembles plans bornés jouissant de cette propriété paradoxale, bien qu'ils ne puissent être* F_σ *et* G_δ *à la fois, ni linéaires; donc, à plus forte raison, ils ne sauraient être fermés, ni ouverts, cependant il y en a qui sont* F_σ *ou* G_δ*. Voilà le sujet principal, mais, à ce propos, j'étudie encore de plus près la notion (bien élémentaire) de* <u>congruence</u> (§3)." And in a footnote he explained, "*Un ensemble est* F_σ*, s'il est une somme dénombrable d'ensembles fermés; s'il est complémentaire d'un* F_σ *(c.-à-d.: produit dénombrable d'ensembles ouverts)—il est* G_δ."

[49] By "Euclidean" is meant, that the distance function is the "natural" or "standard" distance function on $\mathbb{R}^n$—i.e., the square root of the sum of the squares: $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + \cdots}$.

[50] Th 5b for $n = 2$ was generalized by Tarski as theorem **18**(*T*) in [26a], p. 330.

[51] See [26a], pp. 327–329.

[52] If <M, ρ> and <N, σ> are two metric spaces, then a mapping $f: M \to N$ fulfilling $\sigma(f(x), f(y)) \geq \rho(x, y)$ for all $x, y \in M$ is called *expanding*.

4: *Let $B \subset \mathbb{R}$ be a* <u>*bounded*</u> *linear space, and let $A \subseteq \mathbb{R}^n$ for $1 \le n < \infty$ be a subspace of an n-dimensional Euclidean space. If $\delta(A) \ge \delta(B)$, that is, if the diameter of A is not less than the diameter of B,*[53] *and if $f: A \to B$ is an expanding, surjective mapping of A onto B, then f is an isometry between A and B, i.e., $A \cong B$.*

5(L): *Let $A \subset \mathbb{R}^n$ for $1 \le n < \infty$ be a* <u>*bounded*</u> *subspace of an n-dimensional Euclidean space. Then any expanding mapping $f: A \to A$ of the subspace into itself is an isometry between A and $f(A)$, i.e., $A \cong f(A)$.*

The third one, theorem **14**(L), gives a necessary and sufficient condition for a set to be non-monomorphic:

14(L): *A set is not monomorphic iff it has a denumerable non-monomorphic subset.*

The first two of these theorems, particularly **5**(L), seem to have gone unnoticed, or quickly forgotten, because Hans Freudenthal and Witold Hurewicz published a note in *Fundamenta Mathematicae* in 1936 proving a theorem very closely related to **5**(L) using exactly Lindenbaum's methods from [26].[54]

Indeed it can be argued that [26a] went generally unremarked before the Second World War. Stanisław Ruziewicz's review of it in ***JFM*** was perfunctory nearly to the point of dereliction of the reviewer's duty: scarcely 3½ or 4 lines of text, suggesting Ruziewicz himself had only skimmed the work, and offering no reasons why anyone else should want to do even that much. Other than Ruziewicz's review, neither ***Zbl***, ***JFM*** nor ***JSL*** contains any reference to [26a] that would suggest it was studied, cited or worked on by others during the interwar period. This was possibly owing to where it appeared: in ***Sprawozdania z posiedzeń Towarzystwa Naukowego Warszawskiego*** <Minutes of the meetings of the Warsaw Society of Arts and Sciences> as opposed to a weighty mathematics journal. It was only during and (mostly) after the Second World War that, thanks to Sierpiński's diligently filling in almost all of its missing proofs, the work started receiving serious attention and citations. Then in 1958 Sierpiński's monograph ***Cardinal and Ordinal Numbers*** capped these efforts and placed the value of [26a] beyond question.

It is instructive to view Lindenbaum's paper [33a] "*Sur les ensembles localement dénombrables dans l'espace métrique*" in the context of Sierpiński [1933] "*Sur les espaces métriques localement séparables*". Both papers are about local properties—then as now a subject of lively interest in topology. They exchanged ideas, shared preliminary drafts of their manuscripts[55], and published their results in consecutive papers in the same issue of ***Fundamenta Mathematicae***.

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[53] For a given (fixed) distance function ρ, the *diameter* $\delta(X)$ of a set X is the farthest distance between any two points in X—i.e., $\delta(X) = \sup\{\rho(x, y): x, y \in X\}$. Since a distance function is by definition real valued, it is always possible to compare diameters of spaces of different dimension.

[54] See Freudenthal and Hurewicz [1936].

[55] See [33a], p. 102, footnote 10, and p. 104, footnote 13; and Sierpiński [1933], p. 107, footnote 2.

Sierpiński's paper characterized locally denumerable[56] sets in a separable metric space. Alluding to this in [33a] at the bottom of page 101 Lindenbaum wrote, "Or, M. Sierpiński a posé la question *quels sont des ensembles localement dénombrables quand l'espace métrique n'est pas séparable.*" In fact Sierpiński's paper did not explicitly pose this question, only dangled it. Lindenbaum may have meant "m'a posé", i.e., in conversation, or in *marginalia* on a shared manuscript. In any event [33a] took up the question and developed new set-theoretical tools to answer it. "Voici une réponse:" he wrote.

Maurice Fréchet introduced the notion of a separable space in his doctoral dissertation and the importance of the concept was quickly recognized. The definition of a separable metric space can be expressed in various (equivalent) ways. For example, a metric space is said to be *separable*…

iff it has a denumerable open base; or…

iff any open covering of the space admits a denumerable subcovering.

A topological space X is called *separable* iff it contains a denumerable subset D which is dense in X, that is to say, for which $\overline{D} = X$.

A metric space <M, ρ> is said to be *locally separable at a point* $x \in M$ iff there exists an open sphere centred on x which, *qua* subspace, is a separable space. Then a space <M, ρ> is called *locally separable* (simpliciter) iff it is locally separable at every point $x \in M$.

The central result of Sierpiński's paper was its proof of the following "*Théorème: Pour qu'un espace métrique soit localement séparable, il faut et il suffit qu'il soit une somme disjointe d'ensembles ouverts séparables*." <A metric space is locally separable iff it is a disjoint sum of open separable sets.>[57]

Lindenbaum framed his "*réponse*" in brand-new set-theoretical concepts. These are worth spelling out, as it is not generally realized how ground-breaking they were for 1933.

Let ***P*** be a given class of sets, and let <M, ρ> be a given metric space. We will say that a subset $Z \subseteq M$ *is locally* ***P*** (or *has the property* ***P*** *locally*) *at a point* $z \in M$, iff there exists a real number $r > 0$, and a set $Y \in$ ***P***, such that…

$$Y \cap S(z, r) = Z \cap S(z, r)$$

…where $S(z, r)$ is an open sphere of M centred on z with radius r.

Note that for any r satisfying the above condition there is a smaller one that does so too, and thus an infinitely descending sequence of them. This follows from a rudimentary property of open spheres, namely: that for every $p \in S(z, r)$ there is an r' such that $0 < r' < r$ and $S(p, r') \subseteq S(z, r)$.

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[56] We use the word "denumerable" in the sense of "*at most* denumerable", i.e., either finite or at most countably infinite. We understand, for example, that a singleton is denumerable. So is the empty set.

[57] Sierpiński admitted (p.107, footnote 2) that Lindenbaum had pointed out to him that this theorem was "*implicitement contenu dans un théorème de M. Alexandroff* (***Math. Ann.*** 92, p.299, *Fundamentalsatz* 2)", but that Alexandrov's proof was "*plus compliquée que la nôtre*".

Then we can say simply that Z *is locally* $\boldsymbol{P}$ (or *has the property* $\boldsymbol{P}$ *locally*) iff Z is locally $\boldsymbol{P}$ at z for all $z \in Z$. The class of all sets which are locally $\boldsymbol{P}$ will be denoted by $\mathcal{L}(\boldsymbol{P})$.[58]

We say that the set Z is *locally* $\boldsymbol{P}$ *in the restricted sense* iff Z is locally $\boldsymbol{P}$ at z for every $z \in M$. Note that, since $Z \subseteq M$, clearly if Z is locally $\boldsymbol{P}$ in the restricted sense then it is locally $\boldsymbol{P}$ simpliciter. The class of all sets which are locally $\boldsymbol{P}$ in the restricted sense will be denoted by $\mathcal{L}'(\boldsymbol{P})$.

The operators $\mathcal{L}'$ and $\mathcal{L}$ share basic properties with a closure operator; they are…

(1) extensive: $\boldsymbol{P} \subseteq \mathcal{L}'(\boldsymbol{P}) \subseteq \mathcal{L}(\boldsymbol{P})$
(2) isotone: $\boldsymbol{P} \subseteq \boldsymbol{Q}$ implies $\mathcal{L}(\boldsymbol{P}) \subseteq \mathcal{L}(\boldsymbol{Q})$ and $\mathcal{L}'(\boldsymbol{P}) \subseteq \mathcal{L}'(\boldsymbol{Q})$
(3) idempotent: $\mathcal{L}\mathcal{L}(\boldsymbol{P}) = \mathcal{L}(\boldsymbol{P})$ and $\mathcal{L}'\mathcal{L}'(\boldsymbol{P}) = \mathcal{L}'(\boldsymbol{P})$

As an exercise in cardinal arithmetic, one can estimate the cardinalities of $\mathcal{L}(\boldsymbol{P})$ and $\mathcal{L}'(\boldsymbol{P})$. Assume that the space $<M, \rho>$ is infinite. Let $\mathfrak{m}$ be the cardinality of a dense set in M, and $\mathfrak{p}$ be the cardinality of the class $\boldsymbol{P}$. Then the class $\mathcal{L}'(\boldsymbol{P})$ is of cardinality $\leq \mathfrak{p}^{\mathfrak{m}}$. If $\mathfrak{p} > 1$, then the class $\mathcal{L}(\boldsymbol{P})$ is also of cardinality $\leq \mathfrak{p}^{\mathfrak{m}}$.

For a given cardinal number $\mathfrak{n}$, let $\boldsymbol{M}_{\mathfrak{n}}$ be the class of all sets of cardinality $< \mathfrak{n}$ which are contained in M. Thus, for example, $\boldsymbol{M}_2$ consists of the empty set and all singleton subsets of M, while $\boldsymbol{M}_{\aleph_0}$ is the class of all finite subsets of M.

A set Z is said to be *isolated* in a space $<M, \rho>$ if Z and its derived set Z' are disjoint, i.e., if no accumulation point of Z is in Z. It is said to be *divergent* in the space $<M, \rho>$ if it has no accumulation points in M, i.e., if its derived set $Z' = \varnothing$. Every divergent set is isolated; however the converse is not necessarily true.[59] It is easy to see that the elements of $\mathcal{L}(\boldsymbol{M}_2)$ are isolated sets, while $\mathcal{L}'(\boldsymbol{M}_2)$ consists of divergent sets. More interestingly, $\mathcal{L}(\boldsymbol{M}_{\aleph_0}) = \mathcal{L}(\boldsymbol{M}_2)$ and $\mathcal{L}'(\boldsymbol{M}_{\aleph_0}) = \mathcal{L}'(\boldsymbol{M}_2)$.

Since $\boldsymbol{M}_{\aleph_1}$ is the class of all denumerable subsets of M, then $\mathcal{L}(\boldsymbol{M}_{\aleph_1})$ is the class of all *locally denumerable* subsets of M. If M is a separable space, then by definition $\mathcal{L}(\boldsymbol{M}_{\aleph_1}) = \mathcal{L}'(\boldsymbol{M}_{\aleph_1}) = \boldsymbol{M}_{\aleph_1}$.

Lindenbaum's main theorems on locally denumerable sets were as follows:[60]

Th 6. *For a set Z to be locally denumerable, i.e., to belong to* $\mathcal{L}(\boldsymbol{M}_{\aleph_1})$, *it is necessary and sufficient that there exists a sequence of positive real numbers* $\{d_n\}$ ("*distances*") *and a sequence of divergent sets* $\{Z_n\}$ *such that…*

6.1 $Z = \bigcup\{ Z_n : n < \infty\}$; *and…*

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[58] Always bearing in mind, of course, that this definition of $\mathcal{L}(\boldsymbol{P})$ is relative to the given metric space $<M, \rho>$.

[59] We allow ourselves to go off on a small sidetrack here. For readers who may be wondering if *isolated* and *scattered* are the same notion: no, they are not. Every isolated set is scattered, but in general, not all scattered sets are isolated.

[60] Again, using our own numbering scheme, not Lindenbaum's original numbering.

6.2 *if n and m are positive integers, and z an arbitrary element of* Z_n*, then for all points y of* Z_m*, with the possible exception of (at most) one point,* $\rho(x, y) \geq d_m$.

Then by using Sierpiński's result Lindenbaum reformulated the above Th 6 in terms of open sets as:

Th 7. *For a set Z to be locally denumerable it is necessary and sufficient that there exists a class* $\mathbf{G}$ *of open sets such that…*

7.1 *Z is contained in the union of the sets of* $\mathbf{G}$;

7.2 *for every* $G \in \mathbf{G}$*, the set* $Z \cap G$ *is denumerable; and…*

7.3 *if* G_1 *and* G_2 *are distinct elements of* $\mathbf{G}$*, then* $Z \cap G_1 \cap G_2$ *is empty.*

Lindenbaum remarked ([33a], p.106, footnote 18) that the above theorems 6 and 7 (plus several others) remained valid even in a class of spaces—broader than the class of *metric* spaces strictly understood—where there may be no distance at all between distinct points of *M*, i.e., where ρ need satisfy only the conditions…

(Mt 1.1) *if* $x = y$ *then* $\rho(x, y) = 0$ ("half" the law of coincidence); and…

(Mt 2) $\rho(x, y) \leq \rho(z, x) + \rho(y, z)$ (Szymański's modified triangle law).

For a proof of this, he referred the reader to §22 of his doctoral thesis.[61]

This means that Lindenbaum was one of the first, along with E.W. Chittenden (1917) and W.A. Wilson (1931), to broaden or generalize the notion of metric space, and to use this generalized concept as a tool for solving topological problems.[62] Various kinds of generalized metric spaces were considered in later years, e.g., by Karl Menger (1935), Garrett Birkhoff (1936), and Hugo Ribeiro (1943), but none of these authors recognized or remarked on the fact that Lindenbaum had been there before them.

§4. Decomposition of Point Sets, and their Equivalence by Decomposition

Decomposition of point sets and their equivalence by decomposition, i.e., congruence of their respective parts, was a subject of lively interest among the University of Warsaw's mathematicians in the early-to-mid-1920s, when Adolf Lindenbaum entered the university and began studying under them and working with them. Results in 1924 alone included Kuratowski's "*Une propriété des correspondances biunivoques*" <A property of bijections>, Banach's "*Un théorème sur les transformations biunivoques*" <A theorem on bijections>, Tarski's "*O równoważności wielokątów*" <On the equivalence of polygons>, and Banach and Tarski's famous paradox "*Sur la décomposition des ensembles de points en parties respectivement congruentes*" <On the decomposition of point sets into respectively congruent parts>. The first two established some eyebrow-raising facts about one-to-one mappings in

[61] *"…la condition Mt 1.2 n'étant point essentielle [Cf. ma Thèse (Varsovie, 1927; à paraître), §22]."*

[62] Ryszard Engelking, in his treatise *General Topology*, makes extensive use of pseudometric spaces "as a convenient tool" for investigating a wide variety of topological spaces.

purely abstract set-theoretical contexts. The second two were directly about equivalence by decomposition—of polygons in plane geometry, and of point sets in a Euclidean space of finite dimension. It was against this backdrop that Tarski and Lindenbaum set out their results on the theory of cardinal numbers and decompositions of abstract sets in [26a], "*Communication sur les recherches de la théorie des ensembles,*" §2: "*Propriétés des transformations univoques*."

To say that a set A is *decomposed* into a family $\boldsymbol{F}$ of sets means that $\boldsymbol{F}$ is a *partition* on the set A, i.e., $\boldsymbol{F}$ is a family of non-empty disjoint subsets of A whose union $\bigcup\{X : X \in \boldsymbol{F}\}$ is all of A. If $\mathfrak{m}$ is the cardinality of $\boldsymbol{F}$, then A is said to be $\mathfrak{m}$-decomposed, or decomposed into $\mathfrak{m}$ parts. In the same way, A is said to be finitely decomposed, $\aleph_0$-decomposed, $\aleph_1$-decomposed, $2^{\aleph_0}$-decomposed, etc.

Stefan Mazurkiewicz and Wacław Sierpiński had whetted Warsaw's appetite ten years earlier with their 1914 paper "*Sur un ensemble superposable avec chacune de ses deux parties*" <On a set congruent with each of its two parts> wherein they proved that there were nonempty sets A, A_1, $A_2 \subset \mathbb{R}^2$ such that $A = A_1 \cup A_2$, $A_1 \cap A_2 = \varnothing$, $A \cong A_1$, and $A \cong A_2$. What made their result so striking[63] was that the transforms they employed to superpose A_1 and A_2 on A were *rigid*, i.e., they preserved all the "standard" distance relationships on $\mathbb{R}^2$. One of their transforms was a simple rotation through an angle of 1 radian; the other was a straight-line translation over a distance of +1. And they had defined all three sets "effectively", which is to say, without resorting to the axiom of choice, transfinite induction, or the well-ordering theorem; all three sets were denumerable.

Just as their result appeared, however, the First World War broke out, followed by the Soviet westward offensive of 1918–19 and the Polish–Soviet War of 1919–1921, all of which combined to put a damper on Warsaw University's research activities.

In 1921 Stanisław Ruziewicz, working in Lwów, picked up where Mazurkiewicz and Sierpiński had left off, obtaining a related (though not fully analogous) result for a non-denumerable set in $\mathbb{R}^2$: "*Sur un ensemble non dénombrable de points, superposable avec les moitiés de sa partie aliquote*" <On a non-denumerable point set, congruent with halves of its proper subset>.

Using the axiom of choice Ruziewicz defined non-denumerable sets A, B, C, $D \subset \mathbb{R}^2$ such that $A = B \cup C \cup D$, $C \cap D = \varnothing$, and $A \cong C$ and $A \cong D$. The family $\boldsymbol{F} = \{B, C, D\}$ was

[63] On the most obvious level, of course, their result was another example of the paradox of infinity, namely, that the part could equal the whole—a family of paradoxes, actually, with an august lineage, from Zeno of Elea in the 5th century BC, through Galileo's ***Two New Sciences*** of 1638, to Bolzano's ***Paradoxes of the Infinite***, published posthumously in 1851. But after Cantor's work on cardinality, and certainly by 1914, mathematicians had gotten used to such paradoxes. The novelty of Mazurkiewicz and Sierpiński's result lay not in showing that the part could equal the whole (which by then was old news), but in the partition and the transformations they devised, which were truly novel, prefiguring and in a sense anticipating the isometry group $E(n)$ of Euclidean motions and the notion of $E(n)$-equidecomposability, and ultimately the definition of a *paradoxical set*. See also Hausdorff's 1914 paradoxical decomposition of the sphere.

not a strict partition of *A*, only a cover of *A*, as *B* was not necessarily disjoint from $C \cup D$. In fact the set *B* played a rather similar role to the number 0 in Mazurkiewicz and Sierpiński's 1914 proof.[64]

This result of Ruziewicz went some way toward answering (but stopped short of fully answering) a question that Hugo Steinhaus had earlier posed: *Does there exist an uncountable planar set which admits a 2-decomposition each of whose parts is congruent with the whole?*

Lindenbaum went the full distance and answered Steihaus's question in the affirmative in the following theorems:

1. *If A is a linear set congruent with each of two subsets $B \subseteq A$ and $C \subseteq A$, then it is congruent with a subset $D \subseteq (B \cap C) \subseteq A$.*
   (See: [26a], p. 327, theorem **1**(*L*).)

2. *If A is a bounded planar set congruent with each of two subsets $B \subseteq A$ and $C \subseteq A$, then it is congruent with a subset $D \subseteq (B \cap C) \subseteq A$.*
   (*Ibid*., theorem **2**(*L*).)

Cor 1+2: *No linear set, and no bounded planar set, can be decomposed into two parts each of which is congruent with the whole set.*
(See: [26], p. 218, footnote 1.)
Recall of course that Ruziewicz and Sierpiński [1914], and Hausdorff [1914], had shown there *do* exist unbounded planar sets, and sets on the surface of a 3-dimensional sphere, which can be so decomposed.

3. *For every cardinal number $\mathfrak{m} \leq 2^{\aleph_0}$, there exists an unbounded planar set which can be decomposed into $\mathfrak{m}$ parts each of which is congruent with the whole. A similar set can be constructed on the surface of a sphere.*
   (See: [26a], *loc.cit.*, theorem **3***(*L*).)

The above results nicely illustrated how Sierpiński's own later research was shaped by Lindenbaum's. The asterisk* by **3***(*L*) meant that Lindenbaum had used the axiom of choice in his (unpublished) proof. Twenty-one years later Sierpiński [1947b] proved Lindenbaum's third theorem without using the axiom of choice or any of its equivalents,[65] and in so doing answered Steinhaus's question in an effective manner: *There is a constructive proof that, yes, there does exist an uncountable planar set which admits a 2-decomposition each of whose parts is congruent with the whole.* Then Sierpiński went on to supply the missing proofs for

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[64] Ruziewicz actually obtained this result some two years earlier, in the summer of 1919, and ran it past Sierpiński for publication in the first issue of ***Fundamenta Mathematicae*** , i.e., the 1920 issue (Sierpiński was its founding editor). Sierpiński suggested a simplification, which Ruziewicz incorporated. The existence of the set *B* could be proved directly from Zermelo's axioms (including the axiom of choice), without need of Zermelo's theorem on well-ordering, or transfinite numbers. Sierpiński also urged Ruziewicz to consult Hugo Steinhaus, who allegedly had an example of a non-denumerable planar set congruent to its halves. As it turned out, Steinhaus did not have such an example. See the letters from Sierpiński to Ruziewicz dated: 01 April 1919, 17 August 1919, 20 August 1919, and 19 April 1920, in Więsław [2004], pp. 141–143.

[65] Sierpiński wrote: "*La démonstration de A. Lindenbaum n'a pas été publiée et elle m'est inconnue.*" (p. 9)

1, 2, and Cor 1+2 above, which Lindenbaum had said he would deliver "*à plus tard*" but never got around to doing.

Many of the results in [26a] exploited and expanded on Stefan Banach's [1924] "*Un théorème sur les transformations biunivoques*" <A theorem on bijections>, applying it to the theory of functions, cardinal arithmetic and decompositions of point sets.[66] Banach's central theorem could be called "decompositions of pure sets", or the DPS theorem. It stated that:

(**DPS**) *For arbitrary sets A and B, if f is a one-to-one transformation of A onto a subset of B, and g is a one-to-one transformation of a subset of A onto all of B, then there exist decompositions of A and B :* $A = A_1 \cup A_2$, $B = B_1 \cup B_2$, $A_1 \cap A_2 = \varnothing = B_1 \cap B_2$, *such that…* $f(A_1) = B_1$ *and* $g(A_2) = B_2$.

Banach then set out a pair of useful properties which a 2-place relation between sets might possess. He called them property (α), and property (β):

(α) Whenever $A\ R\ B$, there exists a bijection $f: A \to B$ such that for every $X \subseteq A$, $X\ R\ f(X)$.

(β) If $A_1 \cap A_2 = \varnothing = B_1 \cap B_2$, and $A_1\ R\ B_1$, and $A_2\ R\ B_2$, then $(A_1 \cup A_2)\ R\ (B_1 \cup B_2)$.

And with these tools to hand, he proved the following two further theorems:

DPS2 *For a relation R with property* (α), *if A stands in relation R to some subset of B, and some subset of A stands in relation R to B, then there exist decompositions of A and B:* $A = A_1 \cup A_2$, $B = B_1 \cup B_2$, $A_1 \cap A_2 = \varnothing$, $B_1 \cap B_2 = \varnothing$, *such that* $A_1 R\ B_1$ and $A_2 R\ B_2$.

DPS3 *For a relation R with properties* (α) *and* (β), *if A stands in relation R to some subset of B, and some subset of A stands in relation R to B, then A R B.*

Some fundamental relations in set theory and topology turn out to possess properties (α) and (β), e.g., equipollence of pure sets, similarity of ordered sets, homeomorphism of topological spaces, and congruence of point sets. Properties (α) and (β) also crop up (indeed figure prominently) in work on "les types de dimensions", to use Fréchet's expression.[67]

In [26a] Lindenbaum and Tarski used the terms "*relation transformante*" for a relation possessing property (α), and "*relation additive*" for a relation possessing property (β). Two of Lindenbaum's results from [26a] were[68]…

3(*L*) *If* $A \subseteq B \subseteq C$, $A_1 \subseteq C$, *and function* $f: A \to A_1$ *is surjective, then there exist four sets* B_1, D, D_1, *and E such that…*
(a) $A_1 \subseteq B_1 \subseteq C$,
(b) $B = D \cup E$ *and* $B_1 = D_1 \cup E$,

[66] See especially [26a], §2., "*Propriétés des transformations univoques*" <Properties of one-to-one functions>, pp. 316–318.

[67] See Arboleda [1981].

[68] See [26a], page 318.

(c) $D \cap E = \varnothing$ *and* $D_1 \cap E = \varnothing$, *and*

(d) $f(D) = D_1$.

4(*L*) *If* $A \subseteq B \subseteq C$, $A \subseteq C_1$, *and function* $g : C_1 \to C$ *is surjective, then there exist four sets* B_1, *D, E, and* E_1 *such that…*

(a) $A \subseteq B_1 \subseteq C_1$,

(b) $B = D \cup E$ *and* $B_1 = D \cup E_1$,

(c) $D \cap E = \varnothing$ *and* $D \cap E_1 = \varnothing$, *and*

(d) $g(E_1) = E$.

The first of these, **3**(*L*), entailed Banach's DPS theorem, even though it did not require that the function f was one-to-one.

In [26a], §1, "*Théorie des nombres cardinaux*," Lindenbaum used the above results to help him prove **14**(*L*) and **14**(**bis**)[69] relating to the Cantor–Bernstein theorem, which provided Tarski with the machinery he was looking for to derive **15**(*T*).[70] The latter result has since come to be known as Tarski's Mean-value Theorem, and it has its counterpart in theorem **5**(*T*) of §2, p.318, concerning decompositions.

Let *n* be a natural number. Point sets *A* and *B* in a metric space $<M, \rho>$ are said to be *equivalent by n-decomposition*—written $A \equiv_n B$—iff there exist two families of subsets, $\boldsymbol{F}_A = \{A_1, A_2, ..., A_n\}$, and $\boldsymbol{F}_B = \{B_1, B_2, ..., B_n\}$, such that…

$\boldsymbol{F}_A$ is an *n*-decomposition of *A*,

$\boldsymbol{F}_B$ is an *n*-decomposition of *B*, and

A_k is congruent to B_k, i.e., $A_k \cong B_k$, for all $k : 1 \le k \le n$;

…and they are said to be *equivalent by finite decomposition*—written $A \equiv_f B$—iff there exists a natural number n for which $A \equiv_n B$. This can be extended in a natural way to *equivalence by* $\mathfrak{m}$*-decomposition*—written $A \equiv_{\mathfrak{m}} B$—where $\mathfrak{m}$ is an arbitrary transfinite cardinal number.

The above definitions—of equivalence by n-decomposition, by finite decomposition, and by $\mathfrak{m}$-decomposition—are from Banach and Tarski [1924], who established fundamental properties of these relations. Firstly, and most obviously, that for n = 1, $A \equiv_n B$ is simply an isometry $\cong$ in the space $<M, \rho>$. Secondly, that for all $m \ge n$, $A \equiv_n B$ implies $A \equiv_m B$. Thirdly, that for fixed *n*, the relation $\equiv_n$ is not transitive (simple counterexamples suffice to show this). Fourthly, and more interestingly, that equivalence by *finite* decomposition—where you are free to choose a different *n* for each pair of sets—is transitive. Their proof of this used what they called a "double network" method. And since reflexivity and symmetry obviously hold, the relation $\equiv_f$ is an equivalence relation. Moreover, they showed that $\equiv_f$ has Banach properties (α) and (β).

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[69] See pp. 302–303.

[70] *Ibid*, p. 303.

Lindenbaum and Tarski restated most of the above definitions and properties in [26a], along with several new findings (some joint, some by Tarski or Lindenbaum alone):[71] They demonstrated that:

(i) *The relation of equivalence by* $\mathfrak{m}$*-decomposition for* $\mathfrak{m} \geq \aleph_0$ *is completely additive. That is to say, if* $X_1, X_2, ..., X_n, ...$ *and* $Y_1, Y_2, ..., Y_n, ...$ *are two sequences of mutually disjoint sets, and* $X_k \equiv_{\mathfrak{m}} Y_k$ *for all naturals k, then* $(\bigcup_{k=1}^{k<\infty}\{X_k\}) \equiv_{\mathfrak{m}} (\bigcup_{k=1}^{k<\infty}\{Y_k\})$.
(See [26a], p. 328, theorem **6**.)

(ii) *If* $A \subseteq B \subseteq C$ *and* $A \equiv_n C$, *then* $A \equiv_{n+1} B \equiv_{n+1} C$.
(*Ibid*., theorem **9**.)

Lindenbaum supplemented his result from [26], that every bounded linear set was monomorphic, with:

13(*L*). *There exists a bounded linear set A which has a proper subset B such that* $A \equiv_2 B$.
(*Ibid*., page 329.)

He also considered combinatorial properties of congruence and decomposability, as in a result mentioned in [26], p.218, footnote 2:

- *Let n be a natural number, and let A and B be subspaces of a metric space. If* $A \cong B$ *and* $A \cap B$ *contains fewer than* $\frac{n(n+1)}{2}$ *elements, then* $(A - B) \equiv_n (B - A)$.

This was proved for the first time only by Sierpiński [1954], pp.110–113, who also showed, by means of a suitable counter-example on the straight line, that the number $\frac{n(n+1)}{2}$ cannot be any greater.

In his 1943 paper “Some remarks on set theory”[72] Paul Erdős recounted the following interesting story:

> Professor Tarski communicated to me the following result of Lindenbaum: There exist $2^{\mathfrak{c}}$ linear sets no two of which are countable equivalent [by decomposition]. This result was never published, and Tarski does not remember the details of the proof. I have succeeded in proving that if $\mathfrak{m}$ is any cardinal number < $\mathfrak{c}$, then there exist $2^{\mathfrak{c}}$ linear sets no two of which are $\mathfrak{m}$-equivalent. I do not know whether my proof differs from that of Lindenbaum, but I have thought it might be worth publishing, since the result has some interesting applications.[73]

[71] See [26a], §5, “*Théorie des ensembles de points*,” p. 328.

[72] Erdős [1943].

[73] *Ibid*., page 644.

Erdős's proof used the axiom of choice in an essential way. Enter Sierpiński, who four years later was able to prove a generalization—a strengthening—of the same theorem[74] without the axiom of choice, using the same von Neumann function that he and Ruziewicz had used in the 1930s… work which Lindenbaum had surely been aware of at the time. We have to conclude that, quite possibly, Sierpiński's proof merely recapitulated Lindenbaum's original.

From 1931 on, Lindenbaum contributed importantly to investigating Fréchet's "les types de dimensions" of topological spaces.[75]

§5. Decompositions and Equivalence of Polygons in Elementary Geometry

Lindenbaum published only one short note in this area:

[37[a]a] "*Sur l'équivalence de deux figures par décomposition en nombre fini de parties respectivement congruentes*." ***Rocznik Polskiego Towarzystwa Matematycznego*** (= ***Annales de la Société Polonaise de Mathématique***), vol. 16 (année 1937, publ. 1938), p. 197.

This is a (six-line) summary of a lecture given by Lindenbaum on 30 September 1937 to the Third Polish Mathematical Congress in Warsaw. His talk began by setting the problem in its historical perspective, then he outlined recent trends, which he called "quantitatives", then he worked through the proof of a theorem he and Zenon Waraszkiewicz had obtained in 1932, and in conclusion he presented some other related theorems without proofs. We state the Lindenbaum–Waraszkiewicz theorem below. Other details as to what he said in the talk we can only surmise.

Fortunately we know quite a bit about Lindenbaum's work in the theory of equivalence of polygons from two of Tarski's papers: [1931], "*O stopniu równoważności wielokątów*" <On the degree of equivalence of polygons>, and [1931/32], "*Uwagi o stopniu równoważności wielokątów*" <Remarks on the degree of equivalence of polygons>.

In elementary geometry, two polygons are said to be *equivalent* if it is possible to dissect them into the same finite number of respectively congruent polygons having no common *interior* points… though the parts will invariably share common *boundary* points: sides, or segments of sides, or vertices, etc.

This notion has a similar logical structure to the purely set-theoretical notion of equivalence by finite decomposability, $A \equiv_f B$, discussed above. One important difference between them is how they treat boundary points. Another is how they define congruence. The notion of congruent planar figures in elementary geometry is much narrower than the

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[74] See: Sierpiński [1947].

[75] See [31[a]], [34[a]a], [36[a]]; see also Sierpiński [1932a], where Sierpiński quotes verbatim Lindenbaum's two-page proof of a generalization—a strengthening—of one of Sierpiński's own results.

general idea of congruent metric spaces, and even differs in essential ways from the definition of superposable subsets of $\mathbb{R}^2$. Some of the other differences, however, are more superficial: in elementary geometry it is common to use the symbol '$\equiv$' without a subscript, finiteness being taken for granted, and to use verbs like 'divide', 'dissect', or 'cut' (as with scissors) instead of the septic 'decompose', which conjures images of bacterial decay.

To appreciate how big a difference common boundary points make, consider a square *S* with sides of length 1, and an isosceles right triangle *T* with sides of length 2, $\sqrt{2}$ and $\sqrt{2}$. Both *S* and *T* can be cut exactly in half, into mirror-image pairs of isosceles right triangles with sides of length $\sqrt{2}$, 1, and 1. But neither of these divisions is a decomposition, in the set-theoretical sense of a *partition*, because the resulting halves are not disjoint: in both cases they share an edge. And it is not obvious at first sight how to decompose *S* and *T* into respectively disjoint parts such that $S \equiv_f T$. (Sierpiński [1954], p.43, Theorem 15 offers one.)

The following two theorems about equivalence of polygons are provable in the usual axiomatic systems of elementary geometry:

- *If polygon V is a part of polygon W, then these polygons are not equivalent* (known as De Zolt's axiom); and…
- *Polygons V and W are equivalent if and only if they have equal areas* (known as the Wallace–Bolyai–Gerwien theorem).

Tarski was the first to ask if these (or analogous) theorems remain true when equivalence is understood as set-theoretical equivalence $\equiv_f$ by finite decomposition into parts having no common points. Tarski [1924], "*O równoważności wielokątów*" <On the equivalence of polygons>, answered this question in the affirmative using Banach's measure theorem for bounded planar sets (see Banach [1923]).

It is clear that no dissection of equivalent polygons into congruent parts can be unique: two equivalent polygons can always be divided into congruent parts in various ways, with respect to both the form and the number of those parts. Hence the question arises: what is the smallest number of respectively congruent parts that two equivalent polygons can be divided into? For equivalent polygons *W* and *V*, this smallest number is called their *degree of equivalence*, and is denoted by $\sigma(W, V)$.[76] (See Tarski [1931], pp. 37–38.)

Tarski attributed this definition to Lindenbaum in a footnote: "*O ile nam wiadomo, pojęcie to wprowadził Dr. Adolf Lindenbaum (Warszawa), który wraz z autorem artykułu ustalił pewne własności tego pojęcia*" <As far as we know, this notion was introduced by Dr. Adolf Lindenbaum (Warsaw) who together with the author of this article established some properties of the concept>. (See Tarski [1931], p. 38.) Oddly enough, none of the properties or results that he reported in that paper were attributed to Lindenbaum.

[76] The symbol "σ" was chosen because of the first letter of the Polish word "*stopień*", which means "degree".

Tarski defined a function τ as follows: "Let Q be a square with edge a, and let P be a rectangle with edges $x \cdot a$ and a/x, where x is any positive real number. Polygons P and Q are obviously equivalent, and it is easy to see that their degree of equivalence is a function of x; we shall denote this function by the symbol $\tau(x)$. Thus $\tau(x) = \sigma(Q, P)$," …and then he urged his colleagues to join him in investigating its properties.

He himself offered some general results, including upper bounds on $\tau(x)$ for certain values of x. For instance: $\tau(1^1/_3) \leq 3$; $\tau(2^1/_4) \leq 4$; $\tau(n) \leq n$ for all natural numbers n. This last inequality is easy to prove. A square with edge a can be dissected into n mutually congruent rectangular strips with edges a and a/n, from which a rectangle with edges $n \cdot a$ and a/n can be assembled by arranging all the strips end to end.

Tarski conjectured that: (i) $\tau(n) = n$ for every natural number n; and asked whether: (ii) $\tau(x) \geq 3$ for every positive x different from ½, 1 and 2?

In addition to Tarski himself, Henryk Moese, Adolf Lindenbaum, Bronisław Knaster and Zenon Waraszkiewicz investigated the function τ. Moese published a detailed proof of (i) in Moese [1932]. The others' results were published *en masse* in Tarski [1931/32], where it was noted that Zenon Waraszkiewicz was the first to come up with a proof of (i) but did not publish his proof at the time.

Lindenbaum noticed that, "all the results so far obtained for the function $\tau(x)$ still hold when, in the definition of this function, the square is replaced by any rectangle", that is, for a function $\tau_r(x) = \sigma(Q, P)$, where Q is a rectangle with edges a and b, and P is a rectangle with edges $x \cdot a$ and b/x. (See Tarski [1931/32], p. 313) And he also obtained the following:

- If $x = n + {}^1/_p$ where n and p are naturals ≥ 1, then $\tau(x) = n+1$, i.e., $\tau(x)$ rounds x up.
- If $1 < x \leq 2$ then $\tau(x) = 2$ if and only if $x = 1 + {}^1/_p$ where p is a natural ≥ 1.

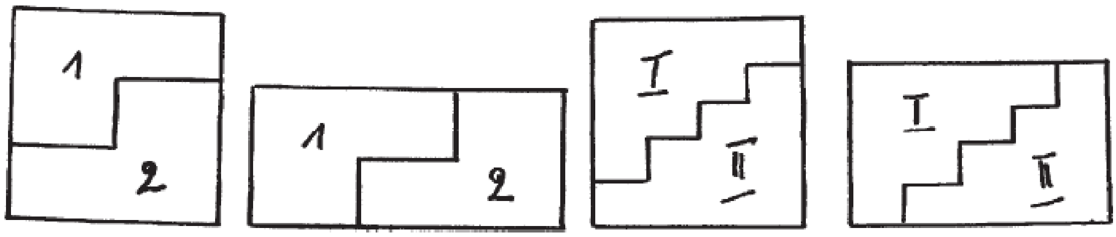
The second of these he obtained jointly with Zenon Waraszkiewicz, and its proof was—in Tarski's words—"somewhat complicated" and required "some subtle methods of reasoning." (See Tarski [1931/32], p. 312) It was this theorem and its proof that Lindenbaum delivered to the Third Polish Mathematical Congress in Warsaw on 30 September 1937. In [37[a]a] it is formulated : *For non-congruent rectangles* $a_1 \times a_2$ *and* $b_1 \times b_2$ *each to be decomposed into* 2 *respectively congruent parts, it is necessary and sufficient that either* a_1/a_2 *or* b_1/b_2 *equal* $k + 1/k$ *for some whole positive* k.

At the end of Tarski [1931/32] there was a set of "exercises" for the ambitious reader. Presumably, all of the problems were known to their respective contributors to be solvable. Lindenbaum offered the following challenge:[77]

- Prove that, if W is a convex figure situated in the plane, then $s(W) \geq \frac{\delta(W) \cdot \omega(W)}{2}$, where $s(W)$, $\delta(W)$ and $\omega(W)$ are respectively the area, diameter and width of W.

[77] *Ibid*., page 314.

Sierpiński, in a letter to Stanisław Ruziewicz dated Warszawa, 14/II/1932 r.,[78] wrote: "*Lindenbaum znalazł ciekawy przykład równoważności przez rozkład kwadratu i prostokąta,*" <Lindenbaum found an interesting example of equivalence by decomposition of a square and a rectangle>, and he drew a sketch in his letter to show Ruziewicz how it worked:



It turned out, the decompositions Sierpiński sketched were crucial to the proof that Lindenbaum and Waraszkiewicz devised.

Note. A deeper and broader presentation of the papers Tarski [1924], Banach–Tarski [1924], Tarski [1932], Tarski [1931/32] and Moese [1932], together with excellent English translations of them, can be found in the book McFarland–McFarland–Smith [2014]. For a survey of related results in the seventy years after Hausdorff''s, Banach's and Tarski's works, see the fascinating monograph Wagon [1993].

§6. Bibliography

Part A: Works by Adolf Lindenbaum

In Part A we have selected the main works of Adolf Lindenbaum. We follow the citation style of the Lindenbaum bibliography in Zygmunt–Purdy [2014]. A work is cited by a two-digit year in square brackets. Where several works appear in the same year, alphabetic suffixes designate their order of appearance. Citations of abstracts and short notes are distinguished by a superscript[a]; reviews by a superscript[r].

Zygmunt–Purdy [2014] includes a fourth sub-part, canvassing Lindenbaum's public lectures, which is omitted here.[79] Also note that its list of seven reviews is here expanded to twenty-two, all of them in ***Zentralblatt für Mathematik***.

I. Papers

[26] Contributions à l'étude de l'espace métrique I. ***Fundamenta Mathematicae***, vol. 8 (1926), pp. 209–222. (The second part of [26] was announced on p. 214, note 1, but

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[78] See Więsław [2004], page 158.

[79] We only mention that since Zygmunt–Purdy [2014] was published another of Lindenbaum's lectures has come to our attention—indeed a whole lecture series, hitherto unknown. We leave this and other new findings for another day.

was never published. See [26a], p. 327, note 1, for a correction.) [JFM 52.0585.01 (E. Pannwitz)].

[26a] Communication sur les recherches de la théorie des ensembles (with A.Tarski). ***Sprawozdania z posiedzeń Towarzystwa Naukowego Warszawskiego***, ***Wydział III Nauk Matematycznych i Przyrodniczych*** (= ***Comptes rendus des séances de la Société des Sciences et des Lettres de Varsovie***, ***Classe III***), vol. 19 (1926), pp. 299–330. [JFM 57.1330.02 (S. Ruziewicz)]
(1) Reprinted in: ***Alfred Tarski***, ***Collected Papers***. ***Volume 1***, ***1921–1934***, ed. by S.R. Givant and R. N. McKenzie, Birkhäuser Verlag: Basel 1986, pp. 173–204.

[30] Remarques sur une question de la méthode axiomatique. ***Fundamenta Mathematicae***, vol. 15 (1930), pp. 313–321. [JFM 56.0488.03 (A. Rosenthal)]

[30a] Sur les opérations d’addition et de multiplication dans les classes d’ensembles (with A. Koźniewski). ***Fundamenta Mathematicae***, vol. 15 (1930), pp. 342–355. [JFM 56.0084.02 (R. Baer)]

[31] Sur les ensembles ordonnés. ***Comptes rendus hebdomadaires des séances de l’Academie des Sciences***, Paris, vol. 192 (1931), pp. 1511–1514. [JFM 57.0091.03 (A. Fraenkel); Zbl 0002.18405 (A. Kolmogoroff)]

[33] Sur les ensembles dans lesquels toutes les équations d’une famille donnée ont un nombre de solutions fixé d’avance. ***Fundamenta Mathematicae***, vol. 20 (1933), pp. 1–29. Errata, p. 287. [JFM 59.0095.02 (W. Hurewicz); Zbl 0006.34001 (B. Knaster)]

[33a] Sur les ensembles localement dénombrables dans l’espace métrique. ***Fundamenta Mathematicae***, vol. 21 (1933), pp. 99–106; Errata p. 295. [JFM 59.0567.04 (G. Aumann); Zbl 0008.08806 (E. Čech)]

[33b] Sur les superpositions des fonctions représentables analytiquement. ***Comptes rendus hebdomadaires des séances de l’Académie des Sciences***, Paris, vol. 196 (1933), pp. 1455–1457. [JFM 59.0267.03 (A. Rosenthal); Zbl 0007.06005 (S. Saks)] (For errata see [34a], p. 13, note 1.)

[34] Z teorii uporządkowania wielokrotnego <Sur la théorie de l’ordre multiple> (Polish with French summary). ***Wiadomości Matematyczne***, vol. 37 (1934), pp. 1–35. [JFM 60.0867.01 (S. Ruziewicz); Zbl 0009.30304 (Th. Motzkin)]

[34a] Sur les superpositions des fonctions représentables analytiquement. ***Fundamenta Mathematicae***, vol. 23 (1934), pp. 15–37; errata, *op. cit.*, p. 304. [JFM 60.0195.02 & JFM 60.0195.03 (A. Rosenthal)]; [Zbl 0009.30502 & Zbl 0010.01403 (S. Saks)].

[35] Miara w geometrji (with E. Szpilrajn) <Measure in geometry. Polish>. ***Świat i życie***. ***Zarys encyklopedyczny współczesnej wiedzy i kultury***, vol. 3, Lwów–Warszawa 1935, pp. 586–595.

[36] Sur la simplicité formelle des notions. ***Actes du Congrès International de Philosophie Scientifique***, ***Sorbonne***, ***Paris 1935***, ***VII Logique***, **Actualités scientifiques et industrielles**, vol. 394 (1936), Hermann & C[ie]: Paris 1936, pp. 29–38. [JFM 62.1050.01 (W. Ackermann); JSL 2 (1937), pp. 55–56 (S. C. Kleene)]

[36a] Über die Beschränktheit der Ausdrucksmittel deduktiver Theorien (with A. Tarski). ***Ergebnisse eines mathematischen Kolloquiums***, vol. 7 (1936), pp. 15–22. [JFM 62.0039.02 (F. Bachmann); Zbl 0014.38602 (A. Schmidt); JSL 1, p. 115–116 (B. Rosser)]
(1) On the limitations of the means of expression of deductive theories. In: A. Tarski, ***Logic***, ***Semantics***, ***Metamathematics***. ***Papers from 1923–1938***. Translated by J.H. Woodger, Clarendon Press: Oxford 1956, pp. 384–392. (Revised English translation of Lindenbaum [36a].)
(2) Sur les limitations des moyens d'expression des théories déductives. In: A. Tarski, ***Logique, sémantique, métamathématique. 1923–1944***, vol. 2, **Philosophie pour l'Âge de la Science**, Librairie Armand Collin: Paris 1974, pp. 109–120. (Revised French translation of Lindenbaum [36a].)
(3) Reprinted in ***Alfred Tarski***, ***Collected Papers***. ***Volume 2***, ***1935–1944***, ed. by S.R. Givant and R. N. McKenzie, Birkhäuser Verlag: Basel 1986, pp. 205–212.
(4) O ograniczeniach środków wyrazu teorii dedukcyjnych. In: A. Tarski, ***Pisma logiczno-filozoficzne. Volume 2***, translated and annotated by Jan Zygmunt, Wydawnictwo Naukowe PWN: Warsaw 2001, pp. 147–157. (Polish translation of [36a](1).)

[38] Über die Unabhängigkeit des Auswahlaxioms und einiger seiner Folgerungen (with A. Mostowski). ***Sprawozdania z posiedzeń Towarzystwa Naukowego Warszawskiego, Wydział III Nauk Matematyczno-Fizycznych*** (= ***Comptes rendus des séances de la Société des Sciences et des Lettres de Varsovie***, ***Classe III***), vol. 31 (1938), pp. 27–32. [JFM 64.0932.01 (Th. Skolem); Zbl 0019.29502, p. 295 (A. Schmidt); JSL 4, pp. 30–31 (A. A. Fraenkel)]
(1) On the independence of the axiom of choice and some of its consequences. In: A. Mostowski, ***Foundational Studies***. ***Selected Works***, vol. 1, edited by K. Kuratowski, W. Marek, L. Pacholski, H. Rasiowa, C. Ryll-Nardzewski, and P. Zbierski. **Studies in Logic and the Foundations of Mathematics**, vol. 93, North-Holland Publishing Company: Amsterdam 1979; and PWN—Polish Scientific Publishers: Warszawa 1979, pp. 70–74. (English translation of Lindenbaum [38a] by M.J. Mączyński.) [Zbl 0425.01021 (E. Mendelson)]

II. Abstracts and Short Notes

[26[a]] Sur l'arithmétique des types ordinaux. ***Rocznik Polskiego Towarzystwa Matematycznego*** (= ***Annales de la Société Polonaise de Mathématique***), vol. 5

(année 1926, publ. 1927), pp. 103–104. (Summary of §3 of [26a], presented at the meeting of the Polish Mathematical Society, Warsaw Section, on 23 April 1926.)

[26[a]a] Sur l'indépendence des notions primitives dans les systèmes mathématiques (with A. Tarski). ***Rocznik Polskiego Towarzystwa Matematycznego*** (= ***Annales de la Société Polonaise de Mathématique***), vol. 5 (année 1926, publ. 1927), pp. 111–113. (Presented at the meeting of the Polish Mathematical Society, Warsaw Section, on 17 December 1926.)
(1) Reprinted in ***Alfred Tarski***, ***Collected Papers***. ***Volume 4***, ***1958–1979***, ed. by S.R. Givant and R.N. McKenzie, Birkhäuser Verlag, Basel 1986, pp. 538–540.

[27[a]] Sur quelques propriétés des fonctions de variable réelle. ***Rocznik Polskiego Towarzystwa Matematycznego*** (= ***Annales de la Société Polonaise de Mathématique***), vol. 6 (année 1927, publ. 1928), pp. 129–130. [JFM 54.0298.01] (Presented at the meeting of the Polish Mathematical Society, Warsaw Section, on 24 October 1926.)

[29[a]] Méthodes mathématiques dans les recherches sur le système de la théorie de déduction. ***Księga Pamiątkowa Pierwszego Polskiego Zjazdu Matematycznego we Lwowie w 1927 roku*** (supplement to ***Annales de la Société Polonaise de Mathématique***), Kraków 1929, p. 36.

[29[a]a] O pewnych własnościach metrycznych mnogości punktowych.—Sur certaines propriétés métriques des ensembles de points. ***Księga Pamiątkowa Pierwszego Polskiego Zjazdu Matematycznego***, ***Lwów***, ***7–10***. ***IX***. ***1927*** (supplement to ***Annales de la Société Polonaise de Mathématique***), Kraków 1929, p. 96.

[31[a]] Sur un ensemble linéaire extrêmement non homogène par rapport aux transformations continues et sur le nombre des invariants de ces transformations. ***Rocznik Polskiego Towarzystwa Matematycznego*** (= ***Annales de la Société Polonaise de Mathématique***), vol. 10 (année 1931, publ. 1932), pp. 113–114. [JFM 58.0648.12] (Presented at the meeting of the Polish Mathematical Society, Warsaw Section, on 16 January 1931.)

[31[a]a] La projection comme transformation continue la plus générale. ***Rocznik Polskiego Towarzystwa Matematycznego*** (= ***Annales de la Société Polonaise de Mathématique***), vol. 10 (année 1931, publ. 1932), pp. 116–117. [JFM 58.0648.13] (Presented at the meeting of the Polish Mathematical Society, Warsaw Section, on 8 May 1931.)

[31[a]b] Sur les figures convexes. ***Rocznik Polskiego Towarzystwa Matematycznego*** (= ***Annales de la Société Polonaise de Mathématique***), vol. 10 (année 1931, publ. 1932), pp. 117–118. [JFM 58. 0805.05] (Presented at the meeting of the Polish Mathematical Society, Warsaw Section, on 8 May 1931.)

[31[a]c] Sur les constructions non-effectives dans l'arithmétique élémentaire. ***Rocznik Polskiego Towarzystwa Matematycznego*** (= ***Annales de la Société Polonaise de Mathématique***), vol. 10 (année 1931, publ. 1932), pp. 118–119. [JFM 58. 1001.14] (Presented at the meeting of the Polish Mathematical Society, Warsaw Section, on 8 May 1931.)

[34[a]] Sur le „problème fondamental" du jeu d'échecs. ***Rocznik Polskiego Towarzystwa Matematycznego*** (= ***Annales de la Société Polonaise de Mathématique***), vol. 13 (année 1934, publ. 1935), pp. 124–125. (Presented at the meeting of the Polish Mathematical Society, Warsaw Section, on 24 February 1933.)

[34[a]a] Sur le nombre des invariants des familles de transformations arbitraires. ***Rocznik Polskiego Towarzystwa Matematycznego*** (= ***Annales de la Société Polonaise de Mathématique***), vol. 13 (année 1934, publ. 1935), p. 131. (Presented at the meeting of the Polish Mathematical Society, Warsaw Section, on 19 January 1934.)

[34[a]b] Remarques sur le groupe des permutations de l'ensemble des nombres entiers. ***Rocznik Polskiego Towarzystwa Matematycznego*** (= ***Annales de la Société Polonaise de Mathématique***), vol. 13 (année 1934, publ. 1935), p. 131. (Presented at the meeting of the Polish Mathematical Society, Warsaw Section, on 19 January 1934.)

[34[a]c] Sur les relations contenues dans les relations ordinales. ***Rocznik Polskiego Towarzystwa Matematycznego*** (= ***Annales de la Société Polonaise de Mathématique***), vol. 13 (année 1934, publ. 1935), p. 132. (Presented at the meeting of the Polish Mathematical Society, Warsaw Section, on 19 January 1934.)

[36[a]] Sur le nombre des invariants des familles de transformations arbitraires, II. ***Rocznik Polskiego Towarzystwa Matematycznego*** (= ***Annales de la Société Polonaise de Mathématique***), vol. 15 (année 1936, publ. 1937), p. 185. (Presented at the meeting of the Polish Mathematical Society, Warsaw Section, on 31 January 1936.)

[37[a]] Numérotage des types logiques. ***Rocznik Polskiego Towarzystwa Matematycznego*** (= ***Annales de la Société Polonaise de Mathématique***), vol. 16 (année 1937, publ. 1938), p. 191. (Summary of a lecture given by Lindenbaum on 30 September 1937 to the Third Polish Mathematical Congress in Warsaw.)

[37[a]a] Sur l'équivalence de deux figures par décomposition en nombre fini de parties respectivement congruentes. ***Rocznik Polskiego Towarzystwa Matematycznego*** (= ***Annales de la Société Polonaise de Mathématique***), vol. 16 (année 1937, publ. 1938), p. 197. (Summary of a lecture given by Lindenbaum on 30 September 1937 to the Third Polish Mathematical Congress in Warsaw.)

[38a] Sur les bases des familles de fonctions. ***Rocznik Polskiego Towarzystwa Matematycznego*** (= ***Annales de la Société Polonaise de Mathématique***), vol. 17 (année 1938, publ. 1939), pp. 124–126.

III. Reviews

[36r] Tsao-Chen Tang, The theorem “p≺q. = .pq = p” and Huntington’s relation between Lewis’s strict implication and Boolean algebra. Bull. Am. Math. Soc. 42 (1936), 743–746. ***Zentralblatt für Mathematik***, vol. 15, no 4 (1936), p. 146. [Zbl 0015.14601]

[36ra] Tsao-Chen Tang, A paradox of Lewis’s strict implication. Bull. Am. Math. Soc. 42 (1936), 707–709. ***Zentralblatt für Mathematik***, vol. 15, no. 4 (1936), p. 146. [Zbl 0015.14602]

[36rb] F. B. Fitch, A system of formal logic without an analogue to the Curry *W* operator. Jour. Symb. Log. 1 (1936), 92–100. ***Zentralblatt für Mathematik***, vol. 15, no. 6 (1936), p. 241. [Zbl 0015.24101]

[36rc] L. Kalmár, Zurückführung des Entscheidungsproblem auf des Fall von Formeln mit einer einzigen, binären, Funktionsvariablen. Compositio Math. 4 (1936), 137–144. ***Zentralblatt für Mathematik***, vol. 15, no. 8 (1936), pp. 338–339. [Zbl 0015.33804]

[37r] W. Hetper, Bases de la sémantique. (Polish) Wiadom. Mat. 43 (1937), 57–86. ***Zentralblatt für Mathematik***, vol. 16, no. 5 (1937), p. 193. [Zbl 0016.19302]

[37ra] Z. Zawirski, Über die Anwendung der mehrwertigen Logik in der empirischen Wissenschaft. Erkenntnis 6 (1937), 430–435. ***Zentralblatt für Mathematik***, vol. 16, no. 5 (1937), p. 195. [Zbl 0016.19502]

[37rb] Jiro Hirano, Zum Zerlegungssatz im erweiterten einstelligen Prädikatenkalkül. Proc. Phys.-Math. Soc. Japan, III, Ser. 19 (1937), 395–412. ***Zentralblatt für Mathematik***, vol. 16, no. 8 (1937), p. 337. [Zbl 0016.33702]

[38r] V. F. von Seckendorff, Beweis des Induktionsschlusses der natürlichen Zahlen aus der Dedekindschen Definition endlicher Mengen. Sitzungsber. Berlin. Math. Ges. 36 (1937), 16–24. ***Zentralblatt für Mathematik***, vol. 17, no. 4 (1938), p. 156. [Zbl 0017.15602]

[38ra] Carl G. Hempel, A purely topological form of non-Aristotelian logic. Jour. Symb. Log. 2 (1937), 97–112. ***Zentralblatt für Mathematik***, vol. 17, no. 6 (1938), p. 241. [Zbl 0017.24102]

[38rb] J.-L. Destouches, Groupe d’équivalence d’une théorie déductive. C. R. Acad. Sci., Paris 205 (1937), 725–727. ***Zentralblatt für Mathematik***, vol. 17, no. 6 (1938), p. 243. [Zbl 0017.24301]

[38^{r}c] L. Kalmár, Zur Reduktion des Entscheidungsproblems. Norsk Mat. Tidsskr. 19 (1937), 121–130. ***Zentralblatt für Mathematik***, vol. 17, no. 8 (1938), p. 337. [Zbl 0017.33702]

[38^{r}d] F. B. Fitch, Modal functions in two-valued logic. J. Symb. Log. 2 (1937), 125–128. ***Zentralblatt für Mathematik***, vol. 17, no. 8 (1938), pp. 337–338. [Zbl 0017.33703]

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Acknowledgments

The authors are immensely grateful for library and archival services provided by the following institutions and digital sources:

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Muzeum Uniwersytetu Warszawskiego; Jan Łukasiewicz fonds
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Apart from persons mentioned in the text, particular thanks for their help in preparing this article are due to:

Bethany J. Antos – The Rockefeller Archive Center
Brigitta Arden – Hillman Library, University of Pittsburgh
Godelieve Bolten – Noord-Hollands Archief, Haarlem (*Wiener Kreis Archief*)
Danuta Bondaryk – Archiwum Państwowe w Warszawie
Frank Bowles – Cambridge University Library
Jacqueline Cox – University of Cambridge, University Archives
Jerzy Dadaczyński – Uniwersytet Papieski Jana Pawła II w Krakowie
Anna Dziedzic – Archiwum Uniwersytetu Warszawskiego
Renata Grzegórska – Archiwum Uniwersytetu Warszawskiego
Colin Harris – University of Oxford, Bodleian Library
Arie Hinkis – Cohn Institute, Tel Aviv University
Danuta Kamińska – Archiwum Państwowe w Warszawie
Brian Keough – SUNY at Albany, M.E. Grenander Dept of Special Collections & Archives

Marek Kietliński – Archiwum Państwowe w Białymstoku
Jean Lambert – CTC & Hughes Hall, University of Cambridge
Magdalena Masłowska – Archiwum Państwowe w Warszawie
Mariusz Pandura – Uniwersytet Wrocławski, Biblioteka Wydziału Nauk Społecznych
Brigitte Parakenings – Bibliothek der Universität Konstanz
Janusz Rudziński – Muzeum Uniwersytetu Warszawskiego
Wojciech Śleszyński – Uniwersytet w Białymstoku
Monika Tomkiewicz – Instytut Pamięci Narodowej. Oddział w Gdańsku
Piotr Wojtylak – Uniwersytet Opolski
Stephen Wordsworth – Council for At-Risk Academics, London South Bank University
Witold Żarnowski – Muzeum Niepodległości w Warszawie

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