Stanisław Jaśkowski and Natural Deduction Systems

Andrzej Indrzejczak

Abstract. In 1934 Stanisław Jaśkowski published his groundbreaking work on natural deduction. At the same year Gerhard Gentzen also published a work on the same topic. We aim at presenting (three versions) of Jaśkowski's system and provide a comparison with Gentzen's approach. We also try to outline the influence of Jaśkowski's approach on the later development of natural deduction systems.

Keywords. Jaśkowski, Gentzen, natural deduction, inclusive logic.

1. Introduction

Stanisław Jaśkowski is one of the founders of modern systems of natural deduction (ND). He presented his system in 1934 as the first volume of the series Studia Logica initiated by J. Łukasiewicz[1]. In fact, ND systems were constructed independently by two logicians; the second was Gerhard Gentzen. It is a matter of coincidence that at the same year Gentzen started to publish his Habilitationschrift which appeared in two parts in *Mathematische Zeitschrift* [14]. The name ND is due to Gentzen – he has called his system Natürliche Kalkül. Jaśkowski used the term "composite system" in contrast to Hilbert axiomatic "simple system"; below we explain the sense of this term in section 3.1.2. Despite the differences in both approaches ND systems were conceived as formal realizations of traditional means of proving theorems in mathematics, science and ordinary discourse. Since then, several variants of ND were devised and presented in hundreds of logic textbooks, giving an evidence that ND systems are commonly accepted as the most efficient way of teaching logic. Still, simplifying a bit, but truly indeed, we may say that everything so far constructed in the field of ND and the related systems is based, more or less directly, on the ideas developed by these two researchers.

[1]This series after the War was revitalised as the well known logical journal.

However, Jaśkowski's role in this enterprise and influence of his results on later developments in the field are not very well recognized. It is a common practice that authors presenting natural deduction systems of one sort or another are mentioning only Gentzen as the inventor of this kind of proof systems. It is not at all surprising that Gentzen's work is widely known since it provides much better developed body of research of a great generality. He did not present just an ND system but also a sequent calculus and provided important theoretical results. His famous normalization theorem for ND was indirectly proved on the basis of equally famous cut elimination theorem holding for sequent calculus. As a byproduct of cut elimination he obtained consistency and decidability results for propositional classical and intuitionistic logic and a version of Herbrand theorem. These profound results of Gentzen are the cornerstones of modern proof theory and this is the main reason that rather modestly looking, relatively short paper of Jaśkowski seems to have gone unnoticed.

But is that true that Jaśkowski's work on ND was really unnoticed and has no impact on the development of research on ND? It is the aim of this paper to show that despite of the absence of Jaśkowski's name in many textbooks presenting ND systems, his solutions had real and strong influence on further research. Roughly, we can say that whereas Gentzen had an enormous impact on the development of theoretical investigations on proof theory, Jaśkowski greatly influenced the practical side of the story. Most of ND systems popularised by hundreds of logic textbooks are directly based on Jaśkowski's proposal, usually without the authors' awareness of the roots of their solutions. We will point out also some other contributions of Jaśkowski's paper, like e.g. introduction of inclusion logic which also passed unnoticed at the time of publication, and were rediscovered by other scholars in later years. Surprisingly the same situation was connected with his invention of the first systems of paraconsistent logics (discursive logics) in 1948. Systematic research on this kind of non-classical logics started in 1960s without the knowledge of Jaśkowski's results.

We start with a general remarks on ND[2], then we provide a detailed description of Jaśkowski's work on ND and compare his solutions with Gentzen's approach. Finally we describe some types of ND which were based on Jaśkowski's solution.

2. Natural Deduction in general

It is a lot of proof systems in use which are called ND systems and sometimes they differ greatly at first sight. Accordingly it is hard to provide a precise definition of ND-systems that would be generally accepted. Some authors tend to use this term in a broad sense, so that it covers also Gentzen's sequent calculus and various forms of tableau systems. In fact, all these systems are in close relationship to each other but we prefer to use this notion in a narrower sense. There are at least

[2]This part is an excerpt from more detailed considerations contained in my [17] and [18].

three reasons to make such a choice. First, for Gentzen his sequent calculus was meant as a technical tool to prove some metatheorems on natural deduction, not as a kind of ND. Secondly, both for him and for Jaśkowski, ND was supposed to reconstruct, in a formally proper way, traditional ways of reasoning. It may be a matter of discusion if existing ND systems realize this task in a satisfying way, but certainly systems like tableaux or resolution are worse in this respect. Finally, taking a term ND in a wide sense would be a classifying operation of doubtful usefulness.

So what is ND in a narrow sense? In Pelletier [25] and Pelletier and Hazen [26] one may find a discussion of several definitions of ND and their inadequacy. Instead of precise definition we provide three criteria which should be satisfied for genuine ND systems:

- ND system allows for entering assumptions into a proof and also for eliminating them.
- ND system consists of rules; there are no (or, at least, very limited set of) axioms.
- ND system admits a lot of freedom in proof construction and possibility of applying several proof search strategies.

Some authors (c.f. [3] or [25, 26]) formulated additional conditions characterising ND, but in our opinion these three are essential. According to this loose characteristics ND system should be open for reasoning from arbitrary assumptions and for the application of different proof constructions. The user is free in constructing direct, indirect, or conditional proofs. He may build more complex formulae or decompose them, as respective introduction/elimination rules allow. Instead of using axioms, or already proved theses, he is rather encouraged to introduce assumptions and derive consequences from them. The presence of axioms is permitted but not essential since their role is taken over by the set of primitive rules. This flexibility of proof construction in ND is in striking contrast to other types of deductive systems usually based on one form of proof.

The above characteristics of ND is still very broad and it allows a lot of freedom in the selection of primitive rules, design of a proof or graphical devices used as bookkeeping devices for indicating the scope of an assumption. These are also important features which make a variety of ND systems presented in textbooks apparently different but are of no real importance in delimiting this class of proof systems. In particular, both Jaśkowski's and Gentzen's approaches were similar in the three points we mentionaed although different in many other respects, and this should be treated as a decisive argument for such a characterization of ND.

Additionaly, in the first ND systems proposed by Jaśkowski and Gentzen one can identify two types of rules which we will call *rules of inference* and *proof construction rules*. The former have the form $\Gamma\ /\ \varphi$; we read them as follows: if we have all formulae from Γ (premises) present in the derivation we can add φ (conclusion) to this derivation. By derivation we mean an attempted, i.e. unfinished proof. *Proof construction rules* are more complex. In general they allow us to build a proof, enter additional assumptions opening nested subderivations, and show under which conditions we may discharge these assumptions and close the respective subderivations. Typical proof construction rules are meant to formalize the old and well known proof techniques like conditional proof, indirect proof, proof by cases e.t.c.

Although much can be said about the prehistory of ND, 1934 is commonly accepted as the first year in the official history of such systems. In this year two groundbreaking papers of Jaśkowski [20] and Gentzen [14] were published. It should be of no surprise that the two logicians with no knowledge of each other's work, independently proposed quite different solutions to the same problem. The need for deduction systems of this sort was in the air. Hilbert's proof theory already offered high standards of precise formalization in terms of axiom systems but the process of actual deduction in Hilbert calculi is usually complicated and needs a lot of invention. Moreover, axiomatic proofs are lengthy, difficult to decipher, and far from informal proofs provided by mathematicains. In consequence, axiom systems, although theoretically satisfying, were considered by many researchers as practically inadequate and artificial. Hence, two goals were involved in this enterprise: a theoretical justification of traditional proof methods on the ground of modern logic, and a formally correct and practically useful system of deduction.

A closer look at the circumstances of Jaśkowski's discovery shows that he may be rightly treated as the first inventor of ND. He was influenced by Łukasiewicz, who posed on his Warsaw seminar in 1926 a problem: how to describe, in a formally proper way, proof methods applied in practice by mathematicians (cf. Woleński [33]). In response to Łukasiewicz problem, Jaśkowski, as a young student[3] presented a first solution in the same year to his tutor. Officially, his first results on ND were announced in 1927, at the First Polish Mathematical Congress in Lvov, mentioned in [19]. Unfortunately, Jaśkowski had a lengthy break in his research due to serious health problems. After recovery in 1932 Jaśkowski gained his doctor's degree under the supervision of Łukasiewicz on the basis of his work on ND. The thesis was eventually published as [20].

3. Jaśkowski's research on ND

Usually two versions of ND are attributed to Jaśkowski, the first called by Pelletier [25] a graphical method and the second a bookkeeping method. We are going to show that it is reasonable to say that Jaśkowski provided three versions of ND, quite similar yet different in a significant way. His first version of ND system (graphical) was not published in 1920s, and we do not know exactly for what logics, in what languages, and by means of what rules, it was conducted. The only thing we know is the format of proof applied by Jaśkowski in the original version since he provided examples in the footnote to [20]. Yet this feature is important enough to treat this proposal as different from the one officially presented in [20].

[3] In 1926 he was 20 years old.

The latter, called by Pelletier a bookkeping method differs significantly at least with respect to proof layout. We will call it the second ND (or the official) system of Jaśkowski.

After the War, Jaśkowski published his lecture notes [21] on mathematical logic in 1947. His presentation of classical logic in the script is not axiomatic but based on the application of ND. It seems that it is the first educational application of ND in the World where adequate system of ND is consequently applied as a form of presentation of classical logic in a textbook. His treatment of ND in [21] is different in some respects from [20] so we feel justified in saying about the third version of Jaśkowski's ND. In what follows we describe in separate sections the second and the third version. Remarks on the first version will be added to the presentation of the second one, because of the lack of knowledge mentioned above.

3.1. Jaśkowski's official ND

Jaśkowski's dissertation is not very long. However, on the 27 pages he provided ND systems for the following logics:

1. positive propositional logic;
2. intuitionistic propositional logic in the version of Kolmogoroff [23];
3. classical propositional logic (CPL);
4. (classical) propositional logic with quantifiers;
5. (inclusive) first-order logic.

Jaśkowski is using a language with $\rightarrow, \neg$ and $\forall$ as primitives, and applies so called Polish notation (parentheses-free) due to Łukasiewicz. In what follows we will be using standard notation for better readability. We do not repeat also the original formulation of rules since it is strongly connected with Jaśkowski's way of displaying proofs in the system and we explain this issue below. Instead we apply some neutral (to the proof format) way of description and additionaly apply $\bot$ (not used as primitive by Jaśkowski, but introduced for illustration) as as a metalinguistic sign of inconsistency.

3.1.1. Rules.

He started with CPL, then he just get rid with negation and a suitable rule for it, corresponding to indirect proof technique. Next, intuitionistic logic is obtained by a slight modification of this rule. So what are the rules for CPL? There are four such rules:

Rule I allows for introduction of an assumption prefixed with the letter 'S' (for supposition) in any place of the proof, hence it is neither inference nor proof construction rule. Rule II – IV formalize (in that order) Conditional Proof, Modus Ponens and (the strong form of) Indirect Proof. So rules II and IV are proof construction rules and III is the only rule of inference in the system. In proof-theoretic formulation (and without specific Jaśkowski's devices) the rules may be described in the following way:

Rule II   If $\Gamma, \varphi \vdash \psi$, then $\Gamma \vdash \varphi \rightarrow \psi$
Rule III $\varphi, \varphi \rightarrow \psi \ / \ \psi$
Rule IV   If $\Gamma, \neg\varphi \vdash \bot$, then $\Gamma \vdash \varphi$

where Γ denotes a set (possibly empty) of other active assumptions. As we mentioned Jaśkowski proposed also modifications of his calculus leading to weaker (he call them incomplete) propositional logics, namely, he has observed that the last rule may be weakened:

Rule IVa   If $\Gamma, \varphi \vdash \bot$, then $\Gamma \vdash \neg\varphi$

This form yields ND formalization of Kolmogoroff's version of intuitionistic logic, whereas deletion of any rule for $\neg$ captures positive logic of Hilbert. It should be underlined that the version of intuitionistic logic considered by Jaśkowski is weaker than the well known Heyting's formalization. In particular, the intuitionistic thesis $\neg p \rightarrow (p \rightarrow q)$ is not provable in his system (although $p \rightarrow (\neg p \rightarrow \neg q)$ is provable). Note that in Gentzen's system this weaker form of indirect proof is sufficient for obtaining Heyting's intuitionistic logic but Gentzen is using $\bot$ as a primitive constant ($\neg$ is definable) and a rule of trivialization $\bot \ / \ \varphi$, so deduction of q from $\neg p$ and p is not a problem. Incidentally, Jaśkowski is also mentioning a theory obtained by addition of $\top$ and $\bot$; we will describe it in connection with the proof format. Also, by the end of his paper he considered proper rules for conjunction and proposed the obvious ones:

$(\wedge I)$ $\varphi, \psi \ / \ \varphi \wedge \psi$
$(\wedge E)$ $\varphi \wedge \psi \ / \ \varphi$ and $\varphi \wedge \psi \ / \ \psi$

Jaśkowski formulated also ND system for propositional logic with universal quantifier, called by him the extended theory of deduction. In such a system we can define $\bot$ as $\forall p, p$ which is in fact shown by Jaśkowski. He defines in an obvious way the notions of free (real) and bound (apparent) propositional variable and add to CPL two new rules which may be formally displayed as follows:

Rule V $\forall p\varphi \ / \ \varphi[p/\psi]$
Rule VI   If $\Gamma \vdash \varphi$, then $\Gamma \vdash \forall p\varphi$

In rule V $\varphi[p/\psi]$ denotes the operation of proper substitution of ψ for p in φ, which means that all occurrences of p which were bound in $\forall p\varphi$ are substituted by ψ and no propositional variable in ψ is bound in $\varphi[p/\psi]$. Rule VI has a side condition that p is not free in any active assumption in Γ. Although we have formulated it as a proof construction rule it may be also described as inference rule with side conditions since there is no subtraction from the set of active assumptions.

Finally Jaśkowski developed ND for first order logic (calculus of functions) but with explicit remark that it is weaker than classical version. Jaśkowski states that "whether individuals exist or not, it is better to solve this problem through other theories. We shall present therefore a system [...] where all theses will be satisfied in the null filed of individuals" (p. 28). Thus he introduced the first system of inclusive logic, whereas the first recognized system of this type was presented in 1950s[4].

Jaśkowski realised that in such a system free variables are in fact not variables but rather (nonlogical) constants whose existence we assume in the proof. Such variables are called valid in the part of a proof where we declared their existence. He did not introduce different kind of letters for denoting free variables (as Gentzen did) but instead he applied a special technique of explicit signification that some variable is held constant for the sake of proof. It is an additional rule which parallels the rule I. This rule VII allows to introduce a term supposition Tx for any variable not valid (so far) in this part of proof. Also rule I must be restricted; we can introduce as assumptions only such formulae which do not contain free variables not valid in the respective part of proof. The remaining rules are variants of rules V and VI:

Rule Va $\quad Ty, \forall x\varphi \;/\; \varphi[x/y]$
Rule VIa $\quad$ If $\Gamma, Tx \vdash \varphi$, then $\Gamma \vdash \forall p\varphi$

Note that $\varphi[x/y]$ denotes the proper substitution of y for x in φ but only if y is valid in respective part of a proof (Ty was introduced earlier by rule VII), hence in contrast to rule V this is a two-premise rule. Also VIa is a "real" proof construction rule since the term-assumption Tx ceased to be valid in the result of its application and it is deleted from the set of active assumptions (a respective subproof is discharged).

3.1.2. Proof format. In the case of ND it is very important how we define a proof since uncontrolled introduction of assumptions without clear indication of their scope may lead to troubles very often met when students are taught to use ND.

Let us anlayse the following "proof" apparently performed with the help of Jaśkowski's rules (with added rule for disjunction introduction ($\vee I$) in line 3):

1	$Sp \vee r \to q$	R. I
2	Sp	R. I
3	$p \vee r$	$2, \vee I$
4	q	1, 3, R. III
5	$p \to q$	2, 4, R. II
6	$(p \to q) \wedge p$	$2, 5, \wedge I$

But $(p \to q) \wedge p$ does not follow logically from $p \vee r \to q$. We cannot apply ($\wedge I$) correctly to formulae from lines 2 and 5 since assumption Sp is not active in this place. The application of rule II in line 5 discharged the assumption 2 and this formula cannot be further used in the proof. This problem is generally

[4] Cf. historical remarks in Bencivenga [4]; in [5] he also pointed out that Jaśkowski's system may be easily changed into a system of free logic.

connected with the application of such rules as II or IV (and VIa) which are not simply inference rules but proof construction rules showing that if some proof is being constructed on the basis of some assumption, then, in the effect, we obtain another proof in which this assumption is not in force. Thus linear proof admitting additional assumptions and means for closing dependant parts of a proof is in fact not a simple sequence of formulae but rather a richer structure containing nested subderivations (subordinate proofs). To avoid scoping difficulties some devices must be used for separating the parts of proof which are in the scope of discharged assumption, hence not available.

All that we know about the first version of Jaśkowski's ND is that he provided a clear indication of the scope of every assumption introduced into a proof. His first original solution to the problem consisted in making boxes for each assumption and dependent part of a proof. Every introduction of an assumption was connected with starting a new box, and this assumption was always put as the first formula in it (he did not applied the prefix "S" for that). An application of any proof construction rule like II or IV was connected with closing a current box, and inferred formula was immediately written down as the next element of outer derivation. He also used an additional rule of repetition to shift a formula from outer open box to the inner; transition in the other direction was of course forbidden. Schematically, the application of both proof construction rules in his propositional system[5] looks like this:

$$
\begin{array}{l}
\Gamma \\
\quad \boxed{\begin{array}{l} \varphi \\ \Gamma' \\ \vdots \\ \psi \end{array}} \\
\varphi \to \psi
\end{array}
\qquad\qquad
\begin{array}{l}
\Gamma \\
\quad \boxed{\begin{array}{l} \neg\varphi \\ \Gamma' \\ \vdots \\ \bot \end{array}} \\
\varphi
\end{array}
$$

On the diagrams possibly empty $\Gamma' \subseteq \Gamma$ refers to formulae obtained by repetition.

In his official version Jaśkowski applied an apparatus of numeric prefixes, instead of boxes. These are finite sequences of natural numbers separated with dots and written before each formula in a proof (except of a thesis). Each time we enter an assumption we extend prefix with additional number (and a prefix S in front of an assumption.). Each application of rules II, IV (IVa) and VIa is connected with subtraction of the last number in the prefix. Application of inference rules, like II or V is admitted only if prefixes of their premises are initial parts, or are identical, to the prefix of the last formula in a proof (prefix of a conclusion must be identical to it). This way Jaśkowski avoided introduction of repetition as a rule.

Application of prefixes instead of boxes may seem as an editorial simplification but in fact it is connected with some philosophical motivations taken from Leśniewski, concerning dynamic nature of deductive system. Jaśkowski thought of

[5]The two examples provided in the footnote in [20] show only propositional proofs.

prefixes as indicators of domains in which formulae with this prefix are valid. Thus formulae with empty prefixes were ordinary theses (valid in every domain), and those with nonempty prefixes are theses relative to some domain in which some suppositions are postulated as valid. Prefixes are then records of dependency of a formula on assumptions in the context of a proof or on the existence of some objects named by term-suppositions. Domain-validity is hereditary with respect to nested subdomains. Thus a formula e.g. $3.2.1.\varphi$ is valid not only in the domain 3.2.1. but also in $3.2.1.1., 3.2.1.5.2$ but not in 3.2. or 3.3.1. Jaśkowski directly pointed out that "Every domain can be considered as a system having its own axioms and constants, though not every domain gives a complete system, much less an interesting one." (p. 14). As an example of "interesting" system he considered the one with added suppositions St (representing $\top$) and $S\neg u$ (interpreted as $\neg\bot$). As the first is introduced with prefix 7. and the second with prefix 7.1. we can consider every formula valid in the domain prefixed with 7.1. as a thesis of a system with $\top$ and $\neg\bot$ being additional axioms. Such a rationale behind using prefixes is also a reason for using a name "composite system" for ND, as it is "composed of many systems"[6]

Below we illustrate both versions with an example of a proof.

1	p	R. I
2	$\neg\neg\neg p$	R. I
3	p	$(1, rep.)$
4	$\neg\neg p$	R. I.
5	$\neg\neg\neg p$	$(2, rep.)$
6	$\neg p$	$4, 5$, R. IV
7	$\neg\neg p$	$3, 6$, R. IV
8	$p \to \neg\neg p$	$1, 7$ R. II

1	$1.Sp$	R. I
2	$1.1.S\neg\neg\neg p$	R. I
3	$1.1.1.S\neg\neg p$	R. I
4	$1.1.\neg p$	$2, 3$, R. IV
5	$1.\neg\neg p$	$1, 4$, R. IV
6	$p \to \neg\neg p$	$1, 5$, R. II

Additionaly we provide an example of proof of the first two theses in his system with quantifiers:

[6] By the way, an innovation introduced by Jaśkowski (i.e. prefixes) may be classified in a different way; we may treat his second version as the first example of ND defined not on formulae but on labelled formulae.

cf 1	$1.S\forall xyAxy$	R. 1
cf 2	$1.1.Tz$	R. VII
cf 3	$1.1.\forall yAzy$	1, R. Va
cf 4	$1.1.Azz$	3, R. Va
cf 5	$1.\forall zAzz$	4, R. VIa
cf 6	$\forall xyAxy \to \forall zAzz$	$1, 5$, R. II
cf 7	$1.1.1.Tv$	R. VI1
cf 8	$1.1.1.Azv$	$3, 7$, R. Va
cf 9	$1.1.\forall vAzv$	$7, 8$, R. VIa
cf 10	$1.\forall zvAzv$	$2, 9$, R. VIa
cf 11	$1.2.Tx$	R. VII
cf 12	$1.2.1.Ty$	R. VII
cf 13	$1.2.1.\forall vAyv$	$10, 12$, R. Va
cf 14	$1.2.1.Ayx$	$11, 13$, R. Va
cf 15	$1.2.\forall yAyx$	$12, 14$, R. VIa
cf 16	$1.\forall xyAyx$	$11, 15$, R. VIa
cf 17	$\forall xyAxy \to \forall xyAyx$	$1, 16$, R. II

Each line contains a successive thesis of a calculus of function (hence 'cf') valid in respective domain. Absolute theses are formulae cf 6 and cf 17. The example illustrates not only the application of rules for inclusive quantifiers but also a dynamics of the system. For example in line 7 a term assumption is introduced not with a prefix 2 but with a prefix 1.1.1, as a continuation of domain 1.1. In fact, a thesis cf 6 is not in itself very interesting and may be seen as an auxiliary result required for proving cf 17 (to be more precise lines 1- 3 are necessary for a proof of cf 17.).

3.1.3. Adequacy. Demonstration of completeness for ND systems is not demanding. If we have at our disposal adequate axiomatic system it is enough to show that all axioms are provable and primitive rules may by simulated in ND. We can also directly prove completeness for ND with respect to semantics in the same way as it is done for axiomatic systems. Indirect results of the first kind are provided in [20] for all ND systems except his inclusive logics since there were neither axiomatic formulation of such a logic nor semantic one.

Usually the problem for ND systems with additional bookkeeping devices, is to prove their soundness, because all this additional machinery must be somehow "translated" either into semantics of suitable logic or into a simpler syntax of axiomatic system. Jaśkowski [20] established some standard form of soundness proof extensively used by many logicians in respective proofs for ND-systems. Shortly, for each prefixed formula we build its development, which is a descending implication with suppositions for each number in the prefix as antecedents and formula itself as the consequent. For example, the development of prefixed formula $i_1....i_n.\varphi$ is $\psi_1 \to (\psi_2 \to \ldots(\psi_n \to \varphi)\ldots)$, where each $\psi_k,\ 1 \leq k \leq n$ is an assumption introduced with addition of successive i_k to already existing prefix, i.e. we have $i_1....i_k.S\psi_k$ above $i_1....i_n.\varphi$ in the proof. Now we may either directly prove that the development of a formula in each line is semantically valid or that it is provable in respective axiomatic system. In the first case we proceed by showing that the first line of a proof is valid ($\varphi \to \varphi$) and that all rules expressed in terms of developments are validity preserving. In the second case we must prove that some formulae are theses of axiomatic systems. Specifically, Jaśkowski has proved for his rules I, II, III (and additionaly IVa) that the development of a formula in each line is a thesis of an axiomatic system of positive (and intuitionistic logic). For CPL he proved a soundness of his ND directly whereas for the system with quantifiers the result is only pointed out. In case of his inclusive logic no such result was possible of course.

This manner of showing soundness for ND systems is commonly applied. There are many variants of this technique but essentially we proceed in such a demonstration by turning formulae of any proof into formulae or a kind of sequents (we add a record of active assumptions), and then by showing that (such modified) rules are validity preserving[7].

3.2. Natural Deduction in Jaśkowski's Lecture Notes

In 1947 Jaśkowski published in mimeographed form his lecture notes "Elements of Mathematical Logic and Methodology of Deductive Sciences" in polish. The book consists of 105 pages and is of great importance for us since Jaśkowski decided to apply in it his natural deduction rules. It is perhaps the first logic textbook where natural deduction is uniformly used as a method for presentation of logic. It is used from the beginning for proving theorems of logic without any reference to axiomatic systems. Moreover, it is applied also in proofs of metalogical results and even truth-functional semantics is introduced via analysis of ND proofs of selected theses.

A system described in [21] deserves the separate treatment since it contains significant differences with the version from [20]. In [21] Jaśkowski presented:

1. classical propositional logic;
2. propositional logic with quantifiers;
3. classical first-order logic;
4. theory of identity in the second order language.

The most important changes in his later approach to ND are the following:

1. Richer language. Jaśkowski introduced $\wedge, \vee, \leftrightarrow$ and $\exists$ as primitive constants and defined introduction and elimination rules for all of them.

2. Omission of nonclassical logics. Only classical logic is presented in lecture notes. In particular, instead of inclusive logic there is a set of rules characterising classical quantifiers. Moreover, the rules are generalised for second order variables and theory of identity is expressed in the extended language.

---

[7] Although this approach is by no means the only one possible. For example, in [17] we proposed a different general strategy of proving soundness for any ND system in Jaśkowski format.

3. Different style of layout for proofs.

3.2.1. Rules. Again the list of rules is opened by the rule for introducing assumptions: any formula may be added with a horizontal bracket above it as an assumption (no prefix 'S' is attached in front of). The rules for $\rightarrow$ and $\neg$ are without changes and the names for them are: $C1$ (implication introduction), $C2$ (MP) and $N1$ (negation elimination). For the remaining connectives we have the following inference rules:

($K1$) $\varphi, \psi \ / \ \varphi \wedge \psi$
($K2$) $\varphi \wedge \psi \ / \ \varphi$ and $\varphi \wedge \psi \ / \ \psi$
($A1$) $\varphi \ / \ \varphi \vee \psi$ and $\psi \ / \ \varphi \vee \psi$
($A2$) $\varphi \vee \psi, \varphi \rightarrow \chi, \psi \rightarrow \chi \ / \ \chi$
($E1$) $\varphi \rightarrow \psi, \psi \rightarrow \varphi \ / \ \varphi \leftrightarrow \psi$
($E2$) $\varphi \leftrightarrow \psi, \varphi \ / \ \psi$ and $\varphi \leftrightarrow \psi, \psi \ / \ \varphi$

In case of propositional logic with quantifiers two rules for $\forall$ are the same as in [20] (now called ($\forall 1$) and ($\forall 2$)) but he added three more rules for $\exists$:

($\exists 1$) $\varphi[p/\psi] \ / \ \exists p\varphi$
($\exists 2$) $\forall p(\varphi \rightarrow \psi), \exists p\varphi \ / \ \exists p\psi$
($\exists 3$) $\exists p\varphi \ / \ \varphi$

The last one is the rule of eliminating vacuous quantification since it has a side condition that p does not occur in φ. In ($\exists 1$) we have of course a proper substitution of ψ for p in the premiss. Note that there is no one elimination rule for $\exists$ in the system. The possible effect is divided into two rule with ($\exists 2$) being of rather mixed character.

The rules for quantifiers in first order logic are identical, the only difference is that individual variables are bounded instead of propositional ones. Hence in particular, introduction of $\forall$ is not a "real" proof construction rule in this system; we add $\forall x$ to some φ only after checking that x is not free in any active assumption. In contrast to rule VIa from [20] there is no closure of a subproof and the rule exactly parallels rule VI. Of course there is also no rule of introduction of term-assumptions in this system.

In both logics with quantifiers there is some innelegancy in the treatment of $\exists$. However, it works and we avoid the problems usually generated in other ND systems where some rule for $\exists$ elimination is postulated[8]. In order to show that the set of rules is complete it is enough to demonstrate as a thesis $\exists xAx \leftrightarrow \neg\forall\neg Ax$. We will show it in the next subsection after characterising proof format.

Jaśkowski extended the application of his rules for quantifiers to second order logic, mainly to develop the theory of identity. In an informal way he describe

[8]The history of successive versions of Copi's ND with numerous mistaken formulation of this rule is particularly instructive – see e.g. Annellis [2].

first the conditions of proper substitution for predicate variables which is necessary for rules ($\forall 1$) and ($\exists 1$). Identity is first characterised by Leibniz condition $\forall A(Ax \leftrightarrow Ay)$ without explicit introduction of a new constant. Then, in the section on definitions, he formulates its definition $\forall xy(x = y \leftrightarrow \forall A(Ax \leftrightarrow Ay))$ and shows that by addition of it as a new assumption we can deduce all characteristic properties of identity. Finally, when discussing axiomatic systems, he provides also axiomatic characterization:

A1 $\forall x, x = x$

A2 $\forall A \forall xy(x = x \to (Ax \to Ay))$

In all these formulae A is a predicate variable.

3.2.2. Proof format. Proofs in [21] are written in linear form but the system of prefixes is absent in this presentation. Instead Jaśkowski is using horizontal brackets over the assumption and under the last formula in a subproof. This solution looks like a simplified version of his first idea of using boxes. In fact, he tends to simplify other things as well; there is no rule of repetition, no prefixes indicating suppositions and even no numeration of lines. Perhaps resignation from prefixes is connected with resignation from Leśniewski's like treatment of deductive system as a dynamic body of theses valid in different domains. Jaśkowski in [21] just provided a series of separate proofs of theses.

Below we provide a proof of the same propositional thesis which was displayed in section 3.1.2. for illustration.

$$\begin{array}{lll}
1 & \overbrace{p} & ass.\\
2 & \overbrace{\neg\neg\neg p} & ass.\\
3 & \underbrace{\overbrace{\neg\neg p}} & ass.\\
4 & \underbrace{\neg p} & 2, 3, N1\\
5 & \underbrace{\neg\neg p} & 1, 4, N1\\
6 & p \to \neg\neg p & 1, 5, C1
\end{array}$$

In order to show the difference between the inclusive rules from [20] and classical rules we provide two proofs of a thesis which is also valid in inclusive logic. For easier comparison we settle the first proof also in Jaśkowski's new (bracketing) style.

1	$\overbrace{\forall x(Ax \to Bx)}$	R. I
2	$\quad\overbrace{\forall xAx}$	R. I
3	$\qquad\overbrace{Ty}$	R. VII.
4	$\qquad Ay \to By$	1, R. Va
5	$\qquad Ay$	2, R. Va
6	$\qquad\underbrace{By}$	$4, 5$, R. III
7	$\quad\underbrace{\forall xBx}$	$3, 6$, R. VIa
8	$\underbrace{\forall xAx \to \forall xBx}$	$2, 7$, R. II
9	$\forall x(Ax \to Bx) \to (\forall xAx \to \forall xBx)$	$1, 8$, R. II

1	$\overbrace{\forall x(Ax \to Bx)}$	$ass.$
2	$\quad\overbrace{\forall xAx}$	$ass.$
3	$\quad Ax \to Bx$	$1, \forall 1$
4	$\quad Ax$	$2, \forall 1$
5	$\quad Bx$	$3, 4, C2$
6	$\quad\underbrace{\forall xBx}$	$5, \forall 2$
7	$\underbrace{\forall xAx \to \forall xBx}$	$2, 6, C1$
8	$\forall x(Ax \to Bx) \to (\forall xAx \to \forall xBx)$	$1, 7, C1$

The selection of rules for $\exists$ may seem doubtfull at first but, in contrast to some other solutions, it has some advantages. All the rules are simple in their form and normal in the sense that premisses logically imply conclusions. In Gentzen's approach the rule for elimination of $\exists$ is a proof construction rule introducing additional subproof. In systems where some inference rule of this kind is provided it is connected with some (sometimes considerably complicated) side conditions (like in Quine's natural deduction [29] or Słupecki and Borkowski's [6] solution).

In order to show that Jaśkowski's characterization of $\exists$ is sufficient it is enough to demonstrate that $\exists xAx \leftrightarrow \neg\forall x\neg Ax$ is derivable (normality of rules yields soundness).

1	$\overbrace{Ax}$	$ass.$
2	$\quad\overbrace{\neg\neg\forall x\neg Ax}$	$ass.$
3	$\quad\forall x\neg Ax$	$2, CPL$
4	$\quad\underbrace{\neg Ax}$	$3, \forall 1$
5	$\underbrace{\neg\forall x\neg Ax}$	$1, 4, N1$
6	$Ax \rightarrow \neg\forall x\neg Ax$	$1, 5, C1$
7	$\forall x(Ax \rightarrow \neg\forall x\neg Ax)$	$6, \forall 2$
8	$\overbrace{\exists x Ax}$	$ass.$
9	$\exists x\neg\forall x\neg Ax$	$7, 8, \exists 2$
10	$\underbrace{\neg\forall x\neg Ax}$	$9, \exists 3$
11	$\exists x Ax \rightarrow \neg\forall x\neg Ax$	$8, 10, C1$

We first prove an auxiliary thesis (in line 7) which is then used as one of the premisses for the application of ($\exists 2$). Also ($\exists 3$) is used in line 10 to eliminate vacuous quantification.

1	$\overbrace{\neg\exists x Ax}$	$ass.$
2	$\quad\overbrace{\neg\neg Ax}$	$ass.$
3	$\quad Ax$	$2, CPL$
4	$\quad\exists x Ax$	$3, \exists 1$
5	$\quad\underbrace{\neg Ax}$	$1, 4, N1$
6	$\underbrace{\forall x\neg Ax}$	$5, \forall 2$
7	$\neg\exists x Ax \rightarrow \forall x\neg Ax$	$2, 6, C1$
8	$\neg\forall x\neg Ax \rightarrow \exists x Ax$	$7, CPL$

In the proof of the converse we applied ($\exists 1$) in line 4 thus showing that all three rules for $\exists$ yield a complete characterization of $\exists$.

4. Other approaches to ND

ND was not also independently proposed by Gentzen but his proposal is widely known, in contrast to Jaśkowski. Before we try to explain why Gentzen is better recognized as a father of ND we briefly describe the most important similarities and differences[9].

When we consider the rules both approaches are very similar. Gentzen also considered classical and intuitionistic logic; the former also in first order case, the latter in Heyting's version (and not restricted to propositional part). In contrast to Jaśkowski he prefers to work with richer language, in particular because he was interested in the philosophical project of syntactic characterization of logical

[9] More detailed comparison of both approaches may be found in Hazen and Pelletier [16].

constants by means of rules later developed in inferentialist programm. The main difference lies in the format of proof chosen by both authors. Gentzen defined proofs as trees labelled with formulae, where leafs are assumptions and the conclusion is put in the root. Transitions between nodes correspond to elementary inferences.

The distinction between ND-systems using tree- or linear-format of proof seems to be not very serious from the theoretical point of view. After all we can redefine every binary tree as a sequence. But in practice the difference is very important because in linear proof we are dealing with formulae, whereas in tree-proofs we are dealing with their concrete occurrences. Since we may use the same formula many times in linear proof, we are forced to introduce some devices for canceling the part of a proof which is in the scope of an assumption already discharged. Otherwise we could "prove" everything, as was illustrated in sec. 3.1.2.

Scoping difficulties do not occur in tree-proofs because we are operating not on formulae but on their single occurrences. Thus premises of any application of a rule must always be displayed directly over the conclusion. Consequently, we cannot use in a proof something which depends on discharged assumptions, because a part of a proof responsible for deduction of a formula must be reproduced above. So Gentzen did not need to bother about technical devices to block nonvalid deductions. Tree format requires less complicated machinery and is very good in representing ready proofs, because the structure of inferential dependencies is readably represented. Moreover, tree proofs are better for proving theoretical results in proof theory. Prawitz [27] proved the profound result on the existence of proofs in normal forms for many ND systems with tree-proofs (Gentzen did it indirectly by showing equivalence with cut-free sequent calculus). No wonder that in works concerned with theoretical investigations this format is very popular (good witness is Negri and von Plato [24].).

But the features of tree proofs that make them so attractive are also the source of some problems. Tree nicely shows the structure of a finished proof but it is hardly suitable for actual derivation. Mental process of proof construction has rather linear structure; we start with assumptions and deduce conclusions until we get the desired goal. Gentzen himself was well aware of this fact, when he wrote that "we are deviating somewhat from the analogy with actual reasoning. This is so, since in actual reasoning we necessarily have (1) a linear sequence of propositions due to the linear ordering of our utterances, and (2) we are accustomed to applying repeatedly a result once it has been obtained, whereas the tree form permits only of a single use of a derived formula." ([14] citation from [32, page 76]) According to Gentzen however, this form of representation is simpler and resulting deviations are inessential.

The choice of linear proof format has also some computational advantages; we can show that for each proof D in tree format we can provide a linear proof D' such that the length of D' is the same or smaller than the length of D. The converse does not hold because in tree proofs we work with occurrences of formulae when in linear proofs we work with formulae themselves. It forces us to repeat many times the same proof-trees if their starting assumptions are used several times. This also shows that linear proofs are better from the practical point of view.

It seems that Jaśkowski decided to use linear format as much closer to actual reasoning, and much more useful for actual proof-search. The applications of ND in logic textbooks are good witnesses of this choice; there is only a few such books using tree format. Majority of them use linear proofs and basically all of them are some variations on the first two solutions introduced by Jaśkowski.

Despite the apparent differences, one thing is common to all variants of Jaśkowski's ND – an essential idea of dividing a proof into separated and partially ordered subproofs. It appears as the most popular solution in hundreds of textbooks where ND-techniques are applied. His first version (graphical), although abandoned by the author himself, is much more popular nowadays. It has many variants but there is always some graphical device added to linear sequence of formulae in a proof. The original format of boxes was used by Kalish and Montague [22]), but with some adjustments which make their system one of the most flexible in practise. In their system each box is preceded with so called 'show-line' which indicates the goal of deduction to be performed inside the box. After closing a box such a show-line is treated as a new formula in the proof. Simplified account, where each assumption is entered with the vertical line which continues until this subproof is in force, is due to Fitch [10], whereas popular system of Copi [8] applies vertical bracketing to closed subproofs. These solutions were repeated in hundreds of textbooks.

It should be noted also that this approach proved especially useful with respect to many nonclassical logics formalized via ND systems. Because parts of proof are separated graphically it is easy to distinguish between different types of subproofs and formulate several kinds of repetition rules with restrictions on the form of formulae which may be shifted to subproofs. One may find ND systems of this kind for modal logics (c.f. Fitting [11], Garson [13], Indrzejczak [17]), relevant logics (Anderson and Belnap [1]) and many others.

The second solution of Jaśkowski is not so popular in ND setting. Borkowski and Słupecki in their ND system from [6] followed this route but with significant simplifications. First of all they treat prefixes as just line-numbers of the proof. They also avoid a proliferation of prefixes since they do not introduce a new prefix for every assumption. Each thesis is analysed in terms of descending implication and all antecedents are written in the same proof level. The proof is ready if the succedent is deduced. For example, if we want to prove a thesis of the form $\varphi_1 \to (\varphi_2 \to \psi)$ we construct a proof looking like this:

$$\begin{array}{lll} 1. & \varphi_1 & ass. \\ 2. & \varphi_2 & ass. \\ & \vdots & \\ n. & \psi & \end{array}$$

instead of Jaśkowski's more complicated form:

1	$1.S\varphi_1$	R. I
2	$1.1.S\varphi_2$	R. I
	$\vdots$	
n	$1.k.\psi$	
$n+1$	$1.\varphi_2 \to \psi$	$2, n$, R. II
$n+2$	$\varphi_1 \to (\varphi_2 \to \psi)$	$1, n+1$, R. II

Of course, if a thesis to be proved is not an implication we must start with indirect assumption and proceed with indirect proof, hence a rule for indirect proof is a primitive one. In fact it is also the only indispensable proof construction rule in the system since in the definition of a proof they allow for introduction of previously proved theses. In consequence, such rules like introduction of implication, or other based on the introduction of additional assumptions, are admissible in their system but in fact we can dispense with subproofs and additional column of numerals for their indication. The problem of elimination rule for $\exists$ is also solved in the original way in their system. They apply the inference rule which implicitly uses skolemization. One may find an extensive applications of their system to logic and set theory in many textbooks written in Polish as well as in English translation [30].

The system of Słupecki and Borkowski is rather not known outside Poland but it was also interestingly applied in the field of automated deduction. In 1970s Andrzej Trybulec started to develop an integrated framework for deduction of theorems in mathematical theories called Mizar. It is basically a computer environment allowing formalization and proof-checking on the basis of rich library. In 1980s Professor Marciszewski initiated a research program concerned with the applicability of Mizar to construct and to check formal proofs in Słupecki and Borkowski system. The program was developed by numerous scholars in many centers, for example in Opole (Wybraniec-Skardowska, Bryniarski) and Lodz (Malinowski, Nowak, Łukowski). It schows a great potential of Mizar and natural deduction system in formalization of logic and formal theories. Proofs in Mizar are constructed similarly as in Słupecki and Borkowski system although some additional devices for users-friendly presentation are added. The present library is based on axioms of set theory in the version due to Tarski-Grothendieck and includes over 1300 articles written by nearly three hundreds of researchers (see www.mizar.org).

There is a kind of ND systems which at first sight may be seen also as a simplification of Jaśkowski's second variant. I mean here a system of Suppes [31] where in each line of a proof we have added a set of numerals of all assumptions active for the formula in question. But the similarity to Jaśkowski's prefixes is apparent in this kind of ND. Lines in Suppes' ND correspond rather to sequents; a set of numerals is a shortcut for antecedent of a sequent. Such a simplification is possible if all rules operate only on succedents of sequents. We do not enter into the details of such solution but it should be noted that such kind of ND is rather a by-product of Gentzen's later paper [15].

Jaśkowski's third system was not known and it is hard to find similar solutions, except perhaps a system presented by Corcoran and Weaver [9]. Here proofs are written down horizontally with subproofs put in brackets. Thus our example proof in Corcoran's style looks like that:

$$[p[\neg\neg\neg p, p[\neg\neg p, \neg\neg\neg p]\neg p]\neg\neg p]p \rightarrow \neg\neg p$$

Our claim that Jaśkowski's lecture notes are perhaps the first consequent textbook application of ND requires some justification. Quine [29] claims that the first textbook applying ND is due to Cooley [7] and was printed in 1942, then reprinted in 1946. In fact, Cooley applies numerous inference rules throughtout the book, however it may be disputable if it is ND system satisfying our three criteria. Conditional proofs based on additional assumptions are only described on pp. 126–140 but not used as the main form of presentation of logic. Moreover, Cooley did not apply any devices for separating subproofs and a rule for elimination of existential quantifier is stated without sufficient restrictions. Hence in our opinion it cannot be treated as a correct system of ND. It seems that the first textbook which consequently applies ND is that of Fitch [10] published in 1952. In Quine's [29] from 1950, ND is also introduced only in three sections as an illustration rather, not as the main proof system. Quine mentioned also some earlier mimeographed notes of himself and of Rosser which applied ND but I had no possibility to check them. A well known textbook of Rosser [28] is using axiomatic system and introduces additional ND-like rules only as a metalogical devices for simplification of axiomatic proofs.

We can conclude our considerations with the following remark concerning Jaśkowski and Gentzen. Both authors laid down the foundations for further investigations on ND but in a slightly different fashion. Jaśkowski seemed to be more concerned with practical aspects of deduction and his general approach, as well as his technical solutions, are of common classroom and textbook use. On the other hand, Gentzen was more theoretically oriented; his investigations led him to profound results in general proof theory.

This is my own evaluation of Jaśkowski's real influence on ND. It is based on the analysis of his texts and easily verifiable. But it should be contrasted with the real knowledge of his achievements and impact on ND. In the earliest applications of ND, like in Quine's or Fitch's book, the origins of the method are known and confirmed. For example, Fitch in foreword claimed that he is using the method of subordinate proofs since 1941 but both Gentzen and Jaśkowski are mentioned as the source of inspiration. Unfortunately, later authors often tend to say about Fitch's ND and forget about Jaśkowski.

References

[1] Anderson, A., R. and N., D. Belnap, *Entailment: the Logic of Relewance and Necessity*, vol I Princeton University Press, Princeton 1975.

[2] Anellis, I. H., 'Forty Years of "Unnatural" Natural Deduction and Quantification. A History of First-Order Systems of Natural Deduction from Gentzen to Copi', *Modern Logic*, 2(2): 113–152, 1991.

[3] Barth E. M., E. C. Krabbe, *From Axiom to Dialogue*, Walter de Gruyter, Berlin 1982.

[4] Bencivenga, E. 'Free Logics', in: D. Gabbay, F. Guenthner (eds.), *Handbook of Philosophical Logic*, vol III, pp. 373–426, Reidel Publishing Company, Dordrecht 1986.

[5] Bencivenga, E. 'Jaśkowski's universally free logic', *Studia Logica*, 102/6:1095–1102, 2014.

[6] Borkowski L., J. Słupecki, 'A Logical System based on rules and its applications in teaching Mathematical Logic', *Studia Logica*, 7: 71–113, 1958.

[7] Cooley, J.C., *A Primer of Formal Logic*, Macmillan 1942.

[8] Copi I. M., *Symbolic Logic*, The Macmillan Company, New York 1954.

[9] Corcoran, J. and G. Weaver, 'Logical consequence in modal logic: Natural Deduction in S5', *Notre Dame Journal of Formal Logic*, 10:370–384, 1969.

[10] Fitch, F., *Symbolic Logic*, Ronald Press Co, New York 1952.

[11] Fitting, M., *Proof Methods for Modal and Intuitionistic Logics*, Reidel, Dordrecht 1983.

[12] Gabbay, D., *LDS - Labelled Deductive Systems*, Clarendon Press, Oxford 1996.

[13] Garson, J.W. *Modal Logic for Philosophers*, Cambridge University Press, Cambridge 2006.

[14] Gentzen, G., 'Untersuchungen über das Logische Schliessen', *Mathematische Zeitschrift* 39:176–210 and 39:405–431, 1934.

[15] Gentzen, G., 'Die Widerspruchsfreiheit der reinen Zahlentheorie', *Mathematische Annalen* 112:493–565, 1936.

[16] Hazen A. P. and F. J. Pelletier, 'Gentzen and Jaśkowski Natural Deduction: Fundamentally Similar but Importantly Different', *Studia Logica*, 102(6):1103–1142, 2014.

[17] Indrzejczak,A., *Natural Deduction, Hybrid Systems and Modal Logics*, Springer, Berlin 2010.

[18] Indrzejczak,A., *Natural Deduction*, in: *Internet Encyclopedia of Philosophy* www.iep.utm.edu/nat-ded/

[19] Jaśkowski, S., 'Teoria dedukcji oparta na dyrektywach założeniowych' (Theory of deduction based on suppositional directives) in: *Księga Pamiątkowa I Polskiego Zjazdu Matematycznego*, Uniwersytet Jagielloński, Kraków 1929.

[20] Jaśkowski, S., 'On the Rules of Suppositions in Formal Logic' *Studia Logica* 1:5–32, 1934. (reprinted in Storrs McCall (ed.) *Polish Logic 1920–1939*, pp. 232-258, Oxford 1967.)

[21] Jaśkowski, S., *Elementy logiki matematycznej i metodologii nauk ścisłych*, (Elements of mathematical logic and methodology of formal sciences) lecture notes, Toruń 1947.

[22] Kalish, D., and R. Montague, *Logic, Techniques of Formal Reasoning*, Harcourt, Brace and World, New York 1964.

[23] Kolmogoroff A. 'O principie tertium non datur', *Matematicesi Sbornik*, 32:651.

[24] Negri, S., and J. von Plato, *Structural Proof Theory*, Cambridge University Press, Cambridge 2001.

[25] Pelletier F. J. 'A Brief History of Natural Deduction', *History and Philosophy of Logic*, 20: 1–31, 1999.

[26] Pelletier F. J. and A. P. Hazen, 'A History of Natural Deduction', in: D. Gabbay, F. J. Pelletier and E. Woods (eds.) *Handbook of the History of Logic* vol 11, 341–414, 2012.

[27] Prawitz, D. *Natural Deduction*, Almqvist and Wiksell, Stockholm 1965.

[28] Rosser J.B., *Logic for Mathematicians*, McGraw Hill, New York 1953.

[29] Quine W. Van O., *Methods of Logic*, Holt, Rinehart and Winston, New York 1950.

[30] Słupecki J., and L. Borkowski, *Elements of Mathematical Logic and Set Theory*, Pergamon Press, Oxford 1967.

[31] Suppes P., *Introduction to Logic*, Van Nostrand, Princeton 1957.

[32] Szabo M. E. *The Collected Papers of Gerhard Gentzen*, North-Holland, Amsterdam 1969.

[33] Woleński J., *Logic and Philosophy in the Lvov-Warsaw School*, Kluwer, Dordrecht 1989.

Andrzej Indrzejczak
Department of Logic
University of Lodz
Lodz
Poland
e-mail: `indrzej@filozof.uni.lodz.pl`