Alfred Tarski—Auxiliary notes on his legacy

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Abstract. The purpose of this article[1] is to highlight a selected few of Alfred Tarski's career achievements. The choice of these achievements is subjective. Section 1 is a general sketch of his life and work, emphasizing his role as researcher, teacher, organizer and founder of a scientific school. Section 2 discusses his contributions to set theory; section 3, to the foundations of geometry and to measure theory. Section 4 looks at his meta-mathematical work, especially the decision problem for formalized theories, and briefly zeros in on a single work, *A Formalization of Set Theory Without Variables* (co-written with Steven R. Givant), which realizes Tarski's life-long program to algebraize logic and the foundations of set theory.

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§l. Life

Alfred Tarski was a logician and a mathematician. He exerted a significant influence on the whole of the twentieth century development of logic and the foundations of mathematics, and through his work in formal semantics and the foundations of logic also on epistemology, the methodology of science, and the philosophy of language. He was a famous representative of the Lwów-Warsaw school and founder of the Berkeley school of logic and the methodology of sciences. He left behind him a rich and wide scientific legacy in logic, metamathematics, semantics, set theory, the foundations of geometry, general algebra and algebraic logic (Boolean algebra, Boolean algebras with operators, relation algebras, cylindric algebras); nearly all of Tarski's output is influenced by the ideas of algebra. Sensitive to the idiomaticity of language, he wrote precisely and clearly, persuasively setting issues in historical and intuitive perspective. He was a charismatic teacher and lecturer. He recognized the role of aesthetics in science, and he drew attention to the "intrinsic charm and prettiness" of theoretical constructions. He was a fierce champion of academic freedom.

Tarski was born in Warsaw on January 14th, 1901, the eldest son of Ignacy aka Izaak Tajtelbaum (often spelled Teitelbaum) and Róża (née Prussak). He had a younger brother Wacław (1903–194?), who was a lawyer. Alfred died on October 27th, 1983, age 82, in

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Berkeley, California, U.S.A. His creative working life spanned more than 60 years, starting in 1921 and continuing in Warsaw up to August, 1939, then in the U.S.A., first on the East Coast, and then, from 1942 to the end of his life, at the University of California, Berkeley.

A: THE WARSAW YEARS. From 1918 to 1924 Tarski was a student in the Faculty of Philosophy at the University of Warsaw. He studied logic, mathematics and philosophy. Despite the interruptions of the Soviet westward offensive of 1918–19 and the Polish–Soviet War of 1919–1921, which intermittently obliged the University to suspend its operations,[2] he managed to fit all his undergraduate and graduate course work, sitting his doctoral exams, and researching, writing and defending his dissertation into those six tumultuous years—while at the same time publishing four journal articles, and, from January of 1924, taking high school teacher training. His thesis advisor was Stanisław Leśniewski, and his Ph.D. thesis was titled *O wyrazie pierwotnym logistyki* <*On the Primitive Term of Logistic*>.

One year later he successfully defended his "habilitation" thesis, securing a "*veniam legendi*" in the philosophy of mathematics and becoming a "*docent*" of the University of Warsaw—licensed to solicit, accept and fulfill commissions to lecture in its name.[3] He gave "*ćwiczenia*" (tutorials) and "*wykłady zlecone*" (commissioned lectures) on set theory, the methodology of mathematics, the foundations of "school" geometry, and the arithmetic of the natural numbers and the reals. On October 1st, 1929, he joined the University's payroll as a "*starszy asystent*" (senior assistant), and on October 1st, 1934, he was promoted to "*adiunkt*" (no exact English equivalent; higher than senior assistant) in Jan Łukasiewicz's Philosophy Seminar. He tried for the position of "*katedra*" (department chair) at the University Jana Kazimierza in Lwów in 1930, and at the University of Poznań in 1937, but in both cases without success.

From 1925 on, he also taught high school mathematics at *Gimnazjum im. Stefana Żeromskiego* in Warsaw. Throughout this period he kept up a frenetic pace of investigation into mathematical logic, semantics, set theory, measure theory, the foundations of geometry, and the teaching of logic and geometry. Arguably, the whole of his life-long output can be traced to these roots.

His work in sentential calculi, methodology of the deductive sciences, cardinal arithmetic, the Axiom of Choice, the definition of truth and more generally semantics brought him recognition and acclaim. During these years in Warsaw he published 16 abstracts and

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[2] Tarski's first university year, 1918–19, was a write-off. Classes were cancelled. Students and faculty signed up for military service. Stanisław Leśniewski, Stefan Mazurkiewicz and Wacław Sierpiński worked with the military on decoding Soviet communications. Tarski performed community service in lieu of military service.

In June, 1920, lectures were again cancelled and students and faculty again volunteered, Tadeusz Kotarbiński and Jan Łukasiewicz among them. This time Tarski served with a military supply and medical unit.

[3] Jacek Jadacki speculates that Tarski's habilitation thesis was the 50-page paper *Sur les ensembles finis*—i.e., [24c]. See J.J. Jadacki (ed.), **Alfred Tarski**: **dedukcja i semantyka** (**déduction et semantique**), Wydawnictwo Naukowe Semper: Warszawa 2003, p. 117. If true, Tarski must have researched and written two dissertations simultaneously.

short notes, 62 longer papers, 3 reviews, 14 exercises[4] and 2 problems, 11 contributions to discussions, and 3 books…

- ***Pojęcie prawdy w językach nauk dedukcyjnych*** *<The Concept of Truth in the Languages of the Deductive Sciences>* (1933).

- ***O logice matematycznej i metodzie dedukcyjnej*** *<On Mathematical Logic and the Deductive Method>* (1936). This has been translated into 12 other languages. Over the years it has had 4 English-language editions.

- A high school geometry textbook.[5]

By 1930 he had already become a leading figure of the Warsaw school of logic and mathematics, collaborating with nearly all of its preeminent members, as well as members of the Lwów school. He co-authored publications with…

- Stefan Banach (on measure theory, decompositions of point sets in metric spaces, and the paradoxical decomposition of the solid sphere).

- Adolf Lindenbaum (on set theory, and the theory of definability).

- Jan Łukasiewicz (the landmark paper *Untersuchungen über den Aussagenkalkül <Investigations into the Sentential Calculus>*, 1930).

- Kazimierz Kuratowski (on projective sets—the *Tarski–Kuratowski algorithm*).

- Wacław Sierpiński (on inaccessible cardinal numbers).

- Andrzej Mostowski (on Boolean rings/algebras with ordered bases).

He participated actively in academic life both at home and abroad. He was a frequent speaker at meetings of the Warsaw Philosophic Society, the Lwów chapter of the Polish Philosophic Society, the Warsaw Mathematics Society, and the Warsaw Scientific Society. He took part in the Polish Philosophy Congresses of 1923, 1927 and 1936, and the Polish Mathematics Congresses of 1927, 1931 and 1937. He also took part in international mathematics and philosophy conferences in Bologna, Italy (1928), Warsaw (1929), Prague (1934), Paris (1935, 1937), and Amersfoort, the Netherlands (1938). He played a principal role in establishing cooperation and working contacts between the Viennese and Warsaw schools. Thanks to his visits to Vienna in 1930 and 1935, and his lectures *Grundlegung der wissenschaftlichen Semantik <On the Semantics of Science>* and *Über den Begriff der logischen Folgerung <On the Concept of Logical Consequence>* given at the first Paris Congress on Scientific Philosophy in 1935, Polish logic exerted a formative influence on the Vienna Circle.

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[4] For English translations see Chapter 12, *Exercises Posed by Tarski*, in A. McFarland, J. McFarland, James T. Smith (eds.), **Alfred Tarski: Early Work in Poland—Geometry and Teaching. With a Bibliographic Supplement**, Birkhäuser: New York 2014, pp. 243–272.

[5] **Geometria dla trzeciej klasy gimnazjalnej**, co-authored with Z. Chwiałkowski and W. Schayer. For an English translation see McFarland–McFarland–Smith [2014], pp. 273–318.

B: AMERICA. On August 21st, 1939 Tarski traveled to the United States to address the 5th International Congress for the Unity of Science, being held September 3–9 at Harvard University in Cambridge, Massachusetts, and to give a lecture tour at several American universities. Owing to the German invasion of Poland on September 1st, 1939, the massive aerial bombardment of Warsaw that began on the same day, and the course of events that immediately followed, leading to the land siege of Warsaw from September 8th through the 28th, Tarski was obliged for his own good to remain in America[6].

Until the summer of 1942 he stayed on the east coast, with temporary appointments at Harvard, CUNY, and the Institute for Advanced Study at Princeton. During this period he forged relationships with Paul Erdös (at Princeton) and J.C.C. McKinsey (in New York City, though not at CUNY—McKinsey was at NYU.) With Erdös he co-authored a paper on fields of sets and large cardinal numbers, published in 1943. With McKinsey he collaborated on algebraic aspects of general topology, and applications of topological methods in intuitionistic and modal logics; their results appeared much later in three jointly authored papers published in 1944–1948. He also renewed contacts with Kurt Gödel (at Princeton) and Rudolf Carnap and W.V.O. Quine (both at Harvard at that time). In 1941 he published what later turned out to be a groundbreaking work in algebraic logic, a paper titled *On the Calculus of Relations*.

In October, 1942, upon securing a lecturer position at the University of California, Berkeley, he relocated to the west coast. In 1948 he was promoted to full professor in the Department of Mathematics at Berkeley, a position which he held to the end of his life.[7]

His first fourteen years at Berkeley, up to 1956, witnessed the realization / achievement of several major scholarly and organizational undertakings which together cemented Tarski's scientific reputation on a world scale. First, he completed and published the results of some important investigations he had begun before the war. These included…

1. The paper *The Semantic Conception of Truth and the Foundation of Semantics* (1944)—a relatively non-technical, or "technical-lite", exposition of his thinking on semantics, addressed to the philosophical community.

2. The monograph ***A Decision Method for Elementary Algebra and Geometry*** (1st edition 1948, 2nd edition 1951), received to great acclaim by the mathematics community.

3. The twin books ***Cardinal Algebras*** (1949) and ***Ordinal Algebras*** (1956). Both books developed results that had been announced without proof in the 1926 omnibus paper *Communication sur les recherches de la théorie des ensembles*, co-authored by Tarski and Adolf Lindenbaum. J.C.C. McKinsey and two of

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[6] …leaving his wife and children in Warsaw. In fact he had little choice, as his ship's return sailing was cancelled.

[7] Although *emeritus* from 1968, he continued teaching until 1973, and continued supervising Ph.D. candidates right up until his death in 1983.

Tarski's very first doctoral students at Berkeley, Louise H. Chin[8] and Bjarni Jónsson, played major roles in editing *Cardinal Algebras*. *Ordinal Algebras* was the collective effort of Tarski, Bjarni Jónsson, and C.C. Chang.

4. The anthology ***Logic, Semantics, Metamathematics: Papers from 1928 to 1938*** (1956), presenting J.H. Woodger's English translations of seventeen of Tarski's works from the 1920s and 1930s on logic, the theory of truth, and the methodology of the deductive sciences. This anthology made Tarski's pre-war output available to the wider world for the first time.

Second, he marked out new lines of inquiry to do with algebraic logic, decidability, and model theory. Algebraic logic quickly gave rise to McKinsey's *The Algebra of Topology* (1944) and *On Closed Elements in Closure Algebras* (1946); Jónsson's *Boolean Algebras with Operators* (Part l, 1951; Part 2, 1952); and saw work begin on the algebraization of quantificational logic with the help of cylindric algebras. The foundations of the theory of cylindric algebras were worked out by Tarski and his students Louise H. Chin and Frederick B. Thompson in the years 1948–1952. Cylindric algebras were extensively researched at Berkeley, and indeed in centers of logical and algebraic research around the world, up to the 1990s.[9]

Tarski summarized his work on decidability in the short book ***Undecidable Theories*** (1953), written in collaboration with Andrzej Mostowski and Raphael M. Robinson. This book is widely considered a masterpiece of the literature on mathematical logic. In the articles *Some Notions and Methods on the Borderline of Algebra and Metamathematics* (1952) and *Contributions to the Theory of Models: I, II, III* (1954–1955), Tarski introduced the basic conceptual apparatus of model theory—a specialized area of set-theoretic semantics for formalized languages and theories—and he showed that model-theoretic techniques are productive in mathematics. He proved, specifically, that the class of representable relation algebras is an equational class, axiomatizable by a set of equations alone—a finding which spurred further investigation of the logical aspects of relation algebras. More generally, one of the tasks model theory was supposed to fulfill was to provide mathematically tractable definitions of metamathematical and metalogical concepts, such as, for instance, "arithmetical class", definability, elementary equivalence, etc. Tarski also played an important role in the formulation of a general definition of "reduced product", and he demonstrated a range of interesting applications of reduced products. In the years 1957–1962 his students and colleagues C.C. Chang, J. Howard Keisler, Dana Scott, Thomas E. Frayne and Anne C. Morel obtained what are now recognized as classic results in model theory, by applying the methods and techniques of reduced products.

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[8] The Mathematics Genealogy Project lists her as Louise Hoy Chin Lim.

[9] For a beautifully written exposition of the first stages of these investigations, and their prehistory, see [61a]. For later developments, see [71[m]] and [85[m]].

Tarski also invented predicate logic with infinitely long expressions (1957) and model theory for infinitary languages. This theory and its application to the study of inaccessible cardinals were developed and elaborated by Carol Ruth Karp, William Porter Hanf, and Dana Stewart Scott. Of these, Hanf was a student of Tarski, Scott a frequent collaborator, and Karp an avid disciple.

Third, he founded a strong center of logic and foundations of mathematics at Berkeley, which by the mid-1950s already enjoyed international renown. From 1952 to 1970 Tarski was director of a program called *Basic Research in the Foundations of Mathematics*. The program covered all the main areas of the foundations of mathematics, and the number of people taking part in it was impressive. The proceedings for the period July 1st, 1959 to June 30th, 1961 list the following topics: model theory, proof theory, infinitary logics, set theory and its foundations, general theory of algebraic and relational structures, algebraic structures in logic, and the foundations of geometry. At least twenty people were listed as participating in the program as researchers, among them the Poles Jerzy Łoś, Wanda Szmielew, Andrzej Ehrenfeucht, and Jan Mycielski.

In 1958 Tarski and his colleagues established *The Group in Logic and the Methodology of Science*, whose mission was, and remains to this day, to foster and promote interdisciplinary doctoral studies leading to a Ph.D. in logic and the philosophy of science. Its idea was (and is) above all about research in mathematical logic in the broadest sense, and its applications to information theory, computability theory, artificial intelligence, methodology of science, philosophy of science, and philosophy of language. Since 1989 the Group has sponsored *The Annual Alfred Tarski Lectures*. Eminent scholars are invited to give lectures on their current areas of interest.

With the aim of strengthening and broadening international collaboration Tarski organized two international symposia at Berkeley: the first, in 1957/58, devoted to the axiomatic method, with special emphasis on its applications to geometry and physics, and the second, in 1963, devoted to model theory. The fruits of these two conferences, beyond of course their stated aim of fostering international cooperation, which they roundly succeeded in achieving, were two substantial volumes of proceedings, co-edited by Tarski: ***The Axiomatic Method*** (1959) and ***The Theory of Models*** (1965). The papers they contain highlight multifarious, subtle and often surprising connections between the axiomatic method and model theory.

In subsequent years Tarski continued working in almost every area that had occupied him previously, but especially universal algebra, equational logic and cylindric algebras, and the foundations of geometry (see §3 below). His chapter zero to ***Cylindric Algebras. Part I*** (1971) is a beautiful monograph on general algebra. He devoted the last years of his life, with the help of Steven Givant, to the book ***A Formalization of Set Theory without Variables***, a work which at last brought to fruition Tarski's life-long program of algebraizing logic and the foundations of set theory.

In 1956–1957 he was president of the International Union of the History and Philosophy of Science. Within the framework of the Union he set up a Division of Logic, Methodology, and Philosophy of Science, whose purpose was to organize international congresses on logic, methodology and philosophy of science. In setting it up Tarski was partly carrying out one of the pre-war aims of the Unity of Science movement.

Tarski guest lectured at numerous universities in America and internationally. He participated in a huge number of conferences, symposia and congresses. He loved traveling, and was curious about the world. He was a scholar who viewed the meaning of his work not as the mere achieving of results in the form of theorems or theories but as part of a communal effort toward the discovery of scientific truth. It was for this reason that he chose to announce so many of his results jointly with his colleagues and students. The Mathematics Genealogy Project shows him as having had 26 students,[10] among them five women. Almost all of them went on to occupy prominent positions in the world of learning (C.C. Chang, Solomon Feferman, Bjarni Jónsson, H. Jerome Keisler, Richard Montague, Andrzej Mostowski, Julia Robinson, Wanda Szmielew, Robert Vaught).

Tarski took a keen interest in cultural and academic life in post-war Poland. Many a Polish logician, philosopher and mathematician benefited from his hospitality and watchful professional care at Berkeley. He visited Poland several times, including…

- For the symposium *Metody infinitystyczne <Infinitistic Methods>*, held in Warsaw in September, 1959, at which he gave a talk *On Predicative Set Theory* (which was never published). On that occasion he also visited Wrocław and gave a lecture at a meeting of the Wrocław chapter of the Polish Philosophic Society titled *Czym są pojęcia logiczne? <What are Logical Notions?>* which was not published until after his death.

- For a Methodology Colloquium on the justification of assertions and decisions (in 1961).

- For a conference on general algebra (in September, 1964).

He was awarded three honorary doctorates: by Pontificia Universidad Católica de Chile (1974), Université d'Aix-Marseille II (1978), and the University of Calgary (1982). In 1966 he was recipient of the Alfred Jurzykowski Foundation's Millennium Award. In 1981 the University of California at Berkeley awarded him its Berkeley Citation.

He was a member of the United States National Academy of Science, the British Academy, and the Royal Netherlands Academy of Arts and Sciences. In 1945 he was appointed correspondent member of the Polish Academy of Learning.[11]

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[10] The Mathematics Genealogy Project defines a "student of Tarski" as someone who was awarded a Ph.D. and whose dissertation listed Tarski as "Advisor 1" or "Advisor 2."

[11] In March, 1949, he was stripped of this distinction, as were other Polish scholars living abroad at that time, for failing to repatriate.

§2. Set Theory

Set theory was one of Tarski's main research interests almost all his professional life,[12] and it was one of his favourite tools for obtaining results in metamathematics, universal algebra and infinitary logic. His first published work, [21], was a term paper he wrote for Leśniewski's seminar while just a third-year student at the University of Warsaw, analysing Cantor's notion of a well-ordered set. One of his last published works, a monograph written jointly with John E. Doner and Andrzej Mostowski which was published in 1978 three years after Mostowski's death, was a meta-mathematical study of the elementary theory of well-ordering.[13] Among other things it established that this theory admits elimination of quantifiers, is axiomatizable by an infinite recursive axiom set, and consequently is decidable. Tarski also co-authored, with Richard M. Montague and Dana S. Scott, a manuscript for a planned book, to have been titled *An Axiomatic Approach to Set Theory*.[14]

Tarski's set-theoretic works focused on three major areas: (a) general or "pure" set theory, (b) connections between set theory, measure theory and Boolean algebras, and (c) presenting aspects of set theory in abstract algebraic calculi. We cite five examples:

1. He developed the arithmetic of cardinal numbers, and he established recursive formulas for the exponentiation of alephs.[15]

2. He studied the Axiom of Choice by looking for equivalents of it, weaker forms of it, and equivalents of its weaker forms, and mapping out logical relations between the Axiom of Choice and other sentences of set theory, such as the Generalized Continuum Hypothesis, and sentences postulating the existence of very large cardinal numbers. Wacław Sierpiński had proposed such a research program in 1918, in his survey paper *L'axiome de M. Zermelo et son rôle dans la Théorie des Ensembles et Analyse*. From the late 1920s until the mid 1950s Tarski and Sierpiński collaborated and competed to populate the program with results.

Tarski established about 30 equivalents of the Axiom of Choice, many of which concerned operations on or relations between cardinal numbers.[16] One of the simplest he found was this proposition:

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[12] He seemed to tire of it for a spell in 1936 when he wrote to Karl Popper, "*Ich arbeite an einer Monographie aus der Mengenlehre, aber es interessiert mich wenig: alte Sachen, mit denen ich mich schon seit Jahren nicht beschäftigt habe.*" It is not clear if "*schon seit Jahren nicht*" was truth or posturing. Or possibly he just meant he had lost interest in writing a survey of established results; he preferred to work on getting new results. Sadly, the monograph he was referring to, *Theorie der eineinendeutigen Abbildungen*, which he was writing jointly with Adolf Lindenbaum and which was to have been "*ein großes mathematisches Buch*", paid with its life. No working drafts ever surfaced, as far as anyone today knows.

[13] Where "elementary theory of well-ordering" is understood as the set of all formulas of first-order predicate calculus that are true in every structure $<U, R>$, where U is a non-empty set and R is a (binary) relation which well-orders the set U.

[14] Montague died in 1971. For reasons which remain unclear, Scott and Tarski ceased work on the manuscript in 1972. It remains unpublished to this day.

[15] See [25] and [30f].

$\mathfrak{m} = \mathfrak{m} \cdot \mathfrak{m}$ *for every infinite cardinal number* $\mathfrak{m}$.

From time to time he posed open questions. In 1924 he asked whether "$\mathfrak{m} = 2 \cdot \mathfrak{m}$" was equivalent to the Axiom of Choice.[17] This was finally answered—in the negative, by Gershon Sageev—only in 1975. The solution had to wait for a new technique of model construction to be invented.

In [26a] Lindenbaum and Tarski stated without proof that the Generalized Continuum Hypothesis (in the version that speaks of "transfinite numbers" rather than alephs) implied the Axiom of Choice.[18] Proofs of this remarkable result were found by others only in 1947 (by Wacław Sierpiński) and 1954 (by Ernst Specker).

In 1939 he established that the Axiom of Choice is a consequence of [one version of] the assertion that inaccessible sets exist:[19]

> *For every set A there exists a set M with the following properties*:
> (i)   *A is equipollent (equinumerous) to a subset of M*;
> (ii)   *the family of subsets of M which are not equipollent to M is equipollent to M*;
> (iii)   *there exists no set B such that the family of all subsets of B is equipollent to M*.

As mentioned above, he also studied weaker versions of the Axiom of Choice, such as the principle of dependent choice,[20] and the Boolean prime ideal theorem.[21]

3. He gave both a philosophical and a formal analysis of the notion of "finite set" that avoided making any use of the Axiom of Choice or the axiom of infinity, and he showed how the arithmetic of natural numbers could be defined in purely set-theoretic terms if finiteness was taken to mean:

> *The set A is finite iff every non-empty family of subsets of A contains a minimal element with respect to inclusion.*[22]

He framed sentences respectively equivalent to the Axiom of Choice and to the Generalized Continuum Hypothesis in terms of finite sets as so defined.[23] In [65[a]] he

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[16] See [24a], [26], [38d], [39b], [48b], [49], [54], [64[a]b].

[17] In [49[m]] cardinals satisfying "$\mathfrak{m}=2\cdot\mathfrak{m}$" were introduced on an abstract level as "idem-multiple" ($a + a = a$) elements of a cardinal algebra.

[18] Adolf Lindenbaum first posed it as an open question. In 1925 Lindenbaum and Tarski jointly proved it, and asserted it without proof in [26a], §1, page 314, theorem 94.

[19] See [39b].

[20] See [48b].

[21] See [54[a]f], [54[a]g], [54[a]h].

[22] See [24c]. The reader should bear in mind that "minimal element" and "least element" are different concepts. Notice that this definition is independent of the notion of a finite natural number.

[23] See [38c], page 163.

announced several results obtainable in "weak" set theories—i.e., set theories without the Axiom of Choice—having to do with D-finite infinite sets and D-finite infinite cardinals.[24]

The existence of such sets and cardinals had been conjectured by Henri Lebesgue (1904) and by Russell and Whitehead ("mediate cardinals" in the terminology of ***Principia Mathematica***, ∗124·61). Tarski proved there was a set *S* of D-finite infinite cardinals that was isomorphic to the set $\mathbb{R}$ of real numbers, under the "natural" ordering of cardinals and reals. Issues of this kind could only have occurred to someone who cared about fundamental things; for whom the notion of a set still required critical analysis. His work on finiteness inspired many other authors. Andrzej Mostowski was the first, with a metamathematical treatise titled *On the Independence of Definitions of Finiteness in a System of Logic* (1938), followed by Azriel Lévy (1958), Arthur L. Rubin, Jean E. Rubin, Erik Ellentuck (1962, 1965, 1968), John K. Truss (1972, 1984) and Agatha C. Walczak-Typke (2005).

4. In a series papers spanning a period of almost 35 years, from 1930 to 1964, he laid the foundations for the theory of large cardinal numbers, including…

- a new definition of strong inaccessibility, and characteristic properties of inaccessible cardinals;[25]

- several formulations of axioms asserting the existence of inaccessible sets (one such axiom cited above);[26]

- characteristic properties of strongly compact, measurable, and weakly compact cardinals, and open questions surrounding them;[27]

- a research program to investigate interconnections between large cardinals and infinitary logic (some of Tarski's students—William Porter Hanf and H. Jerome Keisler among them—obtained important results in this area).

5. He algebraized key aspects of the general theory of sets, having to do with order types, relation types, cardinal and ordinal arithmetics and operations on infinite sequences. He summarized his work in these areas in the books: ***Cardinal Algebras*** [49[m]], and ***Ordinal Algebras*** [56[m]a]. Both books could trace their origins to the 1926 Lindenbaum–Tarski paper *Communication sur les recherches de la théorie des ensembles*, in which Lindenbaum's role had been immense. Tarski conceded that many results "were originally established by Lindenbaum" and were first stated without proof in [26]. He added that it was impossible to convey "an adequate idea of the extent of my indebtedness" to Lindenbaum.

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[24] A set is said to be *Dedekind-finite*, or *D-finite*, iff there is no bijection of the set onto a proper subset of itself (or equivalently, iff every one-to-one mapping of the set into itself is surjective.) A cardinal number is said to be a *Dedekind-finite cardinal*, or a *D-finite cardinal*, iff it is the cardinality of a D-finite set. In weak set theory— without the Axiom of Choice—it can be proved that if a set is finite byTarski's definition then it is also D-finite, but it cannot be proved that if a set is D-finite then it is also finite by Tarski's definition.

     In [49[m]] D-finite cardinals were introduced on an abstract level as *finite* elements of a cardinal algebra.

[25] See the joint paper with Wacław Sierpinski [30a].

[26] See [38a] and [39b].

[27] See [62]; the joint papers [43] and [61b] with Paul Erdös; and the joint paper [64] with H. Jerome Keisler.

Substantial parts of several classic texts on set theory are based on Tarski's results:

- W. Sierpiński's ***Zarys teorii mnogości***, third edition 1928; and ***Leçons sur les nombres transfinis***, 1928, second edition 1950 (cardinal arithmetic; and equivalents of the Axiom of Choice);

- Sierpiński's ***Cardinal and Ordinal Numbers***, 1958, second edition 1965 (equivalents of the Axiom of Choice and proofs of theorems from Tarski–Lindenbaum [26]);

- H. Bachmann, ***Transfinite Zahlen***, 1955, second edition 1967 (equivalents of the Axiom of Choice and inaccessible numbers);

- K. Kuratowski and A. Mostowski's ***Set Theory***, third edition 1976 (the Cantor-Bernstein theorem and its generalizations; Tarski's recursive formulae for exponentiation of alephs; the number of prime ideals in the power-set algebra of an arbitrary set; basic cardinal equivalents of the Axiom of Choice; the exposition of higher types of inaccessible numbers).

Tarski almost singlehandedly steered the development of set theory: dictating what counted as set theory; what were its important questions, results, applications, methods and tools; where were its frontiers; what was worth working on, and why. In the words of Azriel Lévy:[28]

> "Alfred Tarski started contributing to set theory at a time when the Zermelo-Fraenkel axiom system was not yet fully formulated, and as simple a concept as that of the inaccessible cardinal was not yet fully defined. At the end of Tarski's career the basic concepts of the three major areas and tools of modern axiomatic set theory, namely constructibility, large cardinals and forcing, were already clearly defined and were in the midst of rapid successful development. The role of Tarski in this development was somewhat similar to the role of Moses showing his people the way to the Promised Land and leading them along the way, while the actual entry into the Promised Land was done mostly by the next generation. The theory of large cardinals was started mostly by Tarski, and developed mostly by his school. The mathematical logicians of Tarski's school contributed much to the development of forcing, after its discovery by Paul Cohen, and to a lesser extent also to the development of the theory of constructibility, discovered by Kurt Gödel. As in other areas of logic and mathematics Tarski's contribution to set theory cannot be measured by his own results only; Tarski was a source of energy and inspiration to his pupils and collaborators, of which I was fortunate to be one, always confronting them with new problems and pushing them to gain new ground."

§3. Geometry and Measure Theory

Tarski's work in geometry showcases his fascination with metamathematics, axiom systems and the axiomatic method, and it reveals his genius for developing innovative tools and techniques to extract new ore from old seams. He gave serious thought to the choice of

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[28] A. Lévy, *Alfred Tarski's Work in Set Theory*, **The Journal of Symbolic Logic**, vol. 53 (1988), pp. 2–6; p. 2.

primitive concepts, axiom sets and logical bases for various geometries—affine, hyperbolic, elliptical, Euclidean, projective—and, for each geometry, to the independence, economy and strength of its axioms, the independence, definability and predicativity of its notions, and the decidability and completeness of the system as a whole.[29] A volume he co-edited with Leon Henkin and Patrick Suppes, [59[e]], ***The Axiomatic Method***, ***with Special Reference to Geometry and Physics***, underscores geometry's pride of place by its title alone.

Tarski's earliest paper on geometry, [24b], *O równoważności wielokątów* <*On the Equivalence of Polygons*>, was of a different character. It was primarily a study of measure-theoretical properties of point sets in the Euclidean plane, $\mathbb{R}^2$. The paper is important from a historical perspective as it contains:

- One of the first formulations of the definition of equivalence by finite decompositions: "Two geometric figures (thus in particular, two polygons) are called equivalent when it is possible to divide them into the same finite number of respectively congruent *arbitrary* geometric figures not having *any* common points."

- A proof of the Wallace–Bolyai–Gerwien theorem: *two polygons are equivalent iff they have equal areas*. Tarski's proof relied essentially on Banach's theorem on additive measures on the plane.

- The first announcement of the result known as the Banach–Tarski Paradox: "Any two polyhedra are equivalent" or, in an alternative formulation: "Any cube can be divided into a finite number of parts (without common points), which can then be reassembled to form a cube with an edge twice as long".[30]

- The question, "C*zy koło i wielokąt o równych polach są równoważne*?" (Are a circle and a polygon with equal areas equivalent [by finite decompositions]?). This question has since come to be known as "the Tarski Circle Squaring Problem", ...a name with echoes of antiquity.[31]

Jan Mycielski, an authority on paradoxical decompositions, writes:

"The Banach-Tarski paradox is not an inconsistency of mathematics, although it shows that 2 equals 1 in a certain sense. It is proved within the usual system of set theory (using the Axiom of Choice), and we have no reason to doubt the consistency of that system. But it destroys certain naive intuitions about point sets in three-dimensional space $\mathbb{R}^3$. ...

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[29] See [34], [56c], the joint paper [56b] with E.W. Beth, and the joint papers [65a] and [79] with L.W. Szczerba.

[30] The Banach–Tarski Paradox was fully articulated, and its proof elaborated, in [24d], a subsequent paper co-authored with Banach.

[31] Tarski's circle-squaring problem was finally answered in the affirmative in 1990 by Miklós Laczkovich. See: M. Laczkovich, *Equidecomposability and Discrepancy: a Solution to Tarski's Circle-squaring Problem*, ***Journal für die Reine und Angewandte Mathematik***, vol. 404 (1990), pp. 77–117. Laczkovich proved that the circle could be decomposed into no more than 10^{50} different pieces, which could be rearranged to compose a square of equal area. He needed the Axiom of Choice to obtain his decomposition, which was highly non-constructive.

"This theorem is a striking demonstration that the unrestricted concept of a set of points has little to do with the idea of a physical body, and also that, to develop a reasonable theory of areas, volumes, etc., one must limit oneself to more special sets (e.g., Borel sets, or Lebesgue measurable sets)."[32]

In [30] Tarski gave an effective proof—i.e., without using the Axiom of Choice—that for any infinite set E the following two statements were equivalent:

1. *There exists a finitely additive two-valued measure m on the family of all subsets of E such that $m(E) = 1$ and for every finite subset $F \subseteq E$, $m(F) = 0$.*

2. *There exists a maximal ideal in the field of all subsets of E.*

Then he proved that 2 was a theorem of general set theory with the Axiom of Choice added—i.e., that the Axiom of Choice implied 2. His demonstration of a maximal ideal relied on the well-ordering principle and induction on transfinite ordinals. He developed this same technique and line of reasoning further in the papers [39] and [45] on ideals of various types —$\mathfrak{m}$-additive, prime, $\mathfrak{p}$-saturated—in complete fields of sets.[33]

In [29b] and [38g] Tarski explored logical connections between the existence of paradoxical decompositions and the non-existence of some invariant measures. Following von Neumann's terminology, for any arbitrary set E, any subset I of E, and any group G of transformations of E, a function m defined on the power set of E with values in the set of all nonnegative real numbers is said to be an $[E, I, G]$-*measure* iff m is additive, m is invariant under transformations in G, and $m(I) = 1$. Tarski's remarkable result was:

> *In order for an $[E, I, G]$-measure to exist, it is necessary and sufficient that there be no paradoxical decompositions of I relative to G.*

For an algebraic treatment of this and related issues see [49$^{\mathrm{m}}$], ***Cardinal Algebras***, theorems 14.13, 16.8, 16.12 and 16.13.

Tarski's particular achievement in the foundations of geometry was conceiving the idea of a system of geometry based only on first-order logic with identity (i.e., elementary logic) and completely free from any set theoretical assumptions (set theoretical notions, primitive terms, axioms, rules of inference), and showing that such a system was plausible. He called this system *elementary geometry*.

Tarski's axiomatic system of two-dimensional elementary geometry, $\mathcal{E}_2$, has only two specific (or primitive) notions: a ternary predicate constant β for the relation of between-ness and a quaternary predicate constant δ for the equidistance relation. The formula $\beta(x, y, z)$ is read: *y lies between x and z*, while $\delta(x, y, z, u)$ is read: *x is as distant from y as z is from u*.

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[32] J. Mycielski, Review of ***The Banach–Tarski Paradox*** by Stan Wagon, ***The American Mathematical Monthly***, vol. 94, no. 7, pp. 698–700. The quoted passages are from page 698.

[33] Though published six years apart, [39] and [45] were nominally parts I and II of the same two-part paper, and appeared in consecutive issues of ***Fundamenta Mathematicae***, vol. 32, pp. 45–63, and vol. 33, pp. 51–65. The journal's operations were interrupted by the Second World War.

Individual variables represent only points. By contrast, Hilbert's system of geometry has other primitive geometrical notions in addition to points, for instance lines and planes. Tarski limited himself to points by exploiting the fact that figures (configurations of points) dealt with in traditional geometry are uniquely determined by finite numbers of points.

His system $\mathcal{E}_2$ is based on twelve axioms and one axiom schema, each of which is frame solely in primitive and logical terms. Despite avoiding using defined terms, the axioms are relatively short and their meanings are intuitively clear. The axiom schema defines an infinite recursive set of axioms, and plays the role of Dedekind's continuity axiom—or more precisely, it corresponds to Dedekind's continuity axiom restricted to [all and only] sets which are first-order definable in the language of $\mathcal{E}_2$. Relaxing this restriction and going to a second-order language leads directly to the whole of Euclidean geometry. This means that Tarski did successfully separate elementary geometry from full geometry.

To define a model for $\mathcal{E}_2$ Tarski applied the apparatus of his formal semantics: in particular, his definition of truth. Namely, a relational system $\mathcal{M} = <A, B, D>$ is a *model* of $\mathcal{E}_2$ iff…

(i)   *A is an arbitrary non-empty set, and B and D are respectively a ternary and a quaternary relation among elements of A; and*

(ii)   *All the axioms of* $\mathcal{E}_2$ *hold in* $\mathcal{M}$ *if all the variables are assumed to range over elements of A, and the primitive constants* β *and* δ *are understood to denote the relations B and D, respectively.*

The representation theorem reads:

A relational system $\mathcal{M}$ *is a model of* $\mathcal{E}_2$ *iff* $\mathcal{M}$ *is isomorphic with the Cartesian space* $\mathcal{C}_2(\mathcal{F})$ *over some real closed field* $\mathcal{F}$.

Thus the axiom set for $\mathcal{E}_2$ is sound and adequate. Moreover, since the set of its theorems is identical with the set of all true sentences in the Cartesian space $\mathcal{C}_2(\mathbb{R})$ over the field of real numbers, the system $\mathcal{E}_2$ is complete, and consequently decidable. And the so-called "Non-finitisability theorem" says that $\mathcal{E}_2$ is not finitely axiomatizable.

Tarski [59], *What is elementary geometry?*, was first published in [59[e]]: L. Henkin, P. Suppes and A. Tarski (eds.), ***The Axiomatic Method***, ***with Special Reference to Geometry and Physics***, pp. 16–29. It is one of that volume's most highly polished gems. Perhaps it owes its lapidary sheen to thirty-three years of being turned over and over again in Tarski's great tumbler of a head, starting all the way back in 1926.

For a detailed account of the long and fascinating history of Tarski's thinking on his system of elementary geometry see [99].

§4. Decidable and Undecidable Theories

"By a *decision procedure* for a given formalized theory T we understand a method which permits us to decide in each particular case whether a given sentence formulated in the symbolism of T can be proved by means of the devices available in T (or, more generally, can be recognized as valid in T). The *decision problem* for T is the problem of determining whether a decision procedure for T exists (and possibly of exhibiting such a procedure). A theory T is called *decidable* or *undecidable* according as the solution of the decision problem is positive or negative. As is well known, the decision problem is one of the central problems of contemporary metamathematics. Since only few theories turn out to be decidable, most endeavors are directed toward a negative solution." [34]

Although the search for particular decision *procedures* was plainly evident in Ernst Schröder's focus on the "*Lösungsprobleme*" (solution problems) for his relation algebras, an awareness that the decision problem as defined in [53[m]] was a metamathematical issue in its own right emerged later, in the Hilbert school, with its work on the "*Entscheidungsproblem*" (decision-making problem) to do with the decidability of the "restricted functional calculus" —a particular version of first-order predicate logic. In 1936 Alonzo Church and Alan Turing published independent papers showing that, in general, first-order theories were undecidable. Some special cases of first-order theories, such as Presburger arithmetic, were shown to be decidable, but these were more the exception than the rule.

In his 1946 Princeton address[35] Tarski expressed the view that the decision problem was one of the central issues—if not *the* central issue—of metamathematics of the day, noting that, "Hilbert considered the main task of logic to be the construction of a symbolism for use in solving the general decision problem: this was the *raison d'être* of metamathematics." He presented various open questions in logic and mathematics, at that time including Cantor's continuum hypothesis and Hilbert's tenth problem. Then he surveyed contemporary work and known results on the decision problem and proposed a research program on it.

Broadly speaking, Tarski's contributions to the decision problem were:

1. Proving that certain important theories formalized in classical first-order predicate logic were decidable.

2. Laying the foundations for a general method for proofs of undecidability.

3. Proving the undecidability of certain important non-first-order theories.

There are several ways of proving theories decidable—model-theoretic techniques, syntactic methods, and others. One way is simply to reduce the decidability of the theory under question to that of a theory already known to be decidable, either by an embedding, an

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[34] **Undecidable Theories** by Alfred Tarski, in collaboration with Andrzej Mostowski and Raphael M. Robinson, with contributions by Julia Robinson. North-Holland Publishing Co., Amsterdam, 1953. The quoted paragraph is from Chapter I, *A General Method in Proofs of Undecidability*, §I.1. Introduction, page 3.

[35] See H. Sinaceur (ed., with introduction), *Address to the Princeton University Bicentennial Conference on Problems of Mathematics, December 17–19, 1946, by Alfred Tarski*, **The Bulletin of Symbolic Logic**, vol. 6 (2000), pp. 1–44. See also *Odczyt Alfreda Tarskiego na Konferencji o Problemach Matematyki w Princeton, 17 grudnia 1946*, in [01[m]], pp. 396–413.

extension, or a representation. In proofs of decidability, the full-blown theory of recursive functions is often not required. It is enough just to recognize that certain procedures are algorithmic or effectively calculable.[36]

Historically one of the earliest techniques ever used for proving a theory decidable was the method of eliminating quantifiers… a syntactical "process" whose first appearance in the literature was in a paper by Leopold Löwenheim (1915)[37], and which seems to have been inspired by Schröder's work on the *Eliminationsproblem*. The scare quotes around "process" are intended as a warning that quantifier elimination is not a mechanical procedure. Doner and Hodges (1988) gives a beautifully readable précis of how it works: [38]

> Let *T* be a first-order theory. We say that a set *Φ* of formulas is a *set of basic formulas for T* (or more briefly a *basic set*) if, using the axioms of *T*, one can prove that every formula ϕ of the language is equivalent to a Boolean combination ϕ^* of formulas in *Φ* which has only the same free variables as ϕ.
>
> To analyze *T* by the method of quantifier elimination, one would look for a basic set for *T*. Every axiom system has at least one basic set, namely the set of all formulas of the language. But one would try to find a better basic set than that. There is no exact criterion for a "good" basic set, but one would hope for a basic set *Φ* with at least the following three properties. (1) It should be reasonably small and irredundant. (2) Every formula in *Φ* should have some straightforward mathematical meaning. (3) There should be an algorithm for reducing every formula ϕ to its corresponding ϕ^*.
>
> (3) is precise. When we have it, we can claim to have effective quantifier elimination for *T*. (1) and (2) are more a matter of judgment.
>
> In the best cases we have one thing more: (4) an algorithm which tells us, given any basic sentence ψ, either that ψ is provable or that it is refutable from *T*. Given (3) and (4), we have both a completeness proof and a decision procedure for the theory *T*.

To quote Chang and Keisler, "This method applies only to very special theories. Moreover, each time the method is applied to a new theory we must start from scratch… On the other hand, the method is extremely valuable when we want to beat a particular theory into the ground. When it can be carried out, the method of elimination of quantifiers gives a

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[36] For more comments on this see Tarski [48$^{\mathrm{m}}$], note 10.

[37] See: L. Löwenheim, *Über Möglichkeiten im Relativkalkül <On Possibilities in the Calculus of Relations>*, **Mathematische Annalen**, vol. 76 (1915), pp. 447–470.

[38] The four paragraphs in the box are an extended quote from: John Doner and Wilfrid Hodges, *Alfred Tarski and Decidable Theories*, **The Journal of Symbolic Logic**, Vol. 53, No. 1, (March, 1988), pp. 20–35. The text reproduced here is from page 24 of their article, where they present it in three paragraphs, not four.

tremendous amount of information about a theory … [it] may be thought of as a direct attack on a theory."[39]

Three things should be emphasized. First, quantifier elimination is not just, or even primarily, about proving a theory decidable. It is about discovery generally. Second, it can be applied to all formulas, not just to sentences. Free variables are welcome. Third, it is more art than science. But when it works, it works a treat.

Tarski encountered the method as a student and applied it in his seminar exercises when he was a *docent*. One of the exercises he set Mojżesz Presburger was to use the method of quantifier elimination to investigate the completeness of the arithmetic of integers with addition.[40]

Using the method of elimination of quantifiers, Tarski contributed to a deeper understanding of the first-order theories of dense orders and linear orders.[41] He proved, among other things, that every sentence in the elementary theory of dense order, T_{DO}, was deductively equivalent to a Boolean combination of the following two (basic) sentences: "There is no first element", and "There is no last element". Since the elimination of quantifiers was "effective", in Doner and Hodges's precise sense (3) above, T_{DO} turned out to be decidable. As a corollary, Tarski obtained a classification of all axiomatic extensions of T_{DO}, as well as a semantic characterization of all complete extensions of T_{DO}. There are four such complete extensions—the theories of the order types η, $\eta+1$, $1+\eta$, and $1+\eta+1$.[42]

Together with Andrzej Mostowski and John Elliott Doner, Tarski proved that the elementary theory of well-ordering was decidable, described the ordinals definable by first-order formulas, and gave a description and classification of all models of the theory.[43]

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[39] See: C.C. Chang and H.J. Keisler, **Model Theory**, North-Holland: Amsterdam 1973 (3rd edition, 1990), pages 49–60. This quotation is from page 49 in the 3rd edition.

[40] See: M. Presburger, *Über die Vollständigkeit eines gewissen Systems der Arithmetik ganzer Zahlen, in welchem die Addition als einzige Operation hervortritt*, **Sprawozdanie z I Kongresu Matematików Krajów Słowiańskich, Warszawa 1929** (= **Comptes-rendus du I Congrès des Mathématiciens des Pays Slaves**), Księżnica Atlas: Warszawa 1930, pp. 92–101; see p. 97, footnote 1.
  See also: W. Hodges, *A Visit to Tarski's Seminar on Elimination of Quantifiers*, in J. van Benthem et al. (eds.), **Proof, Computation and Agency**, **Synthese Library** 352, Springer Science+Business Media B.V. 2011; pp. 53–66.

[41] In [36d], §5, Tarski axiomatized these two theories, calling them, respectively, "*die elementare Theorie der dichten Anordnung*" (p.290), and "*die elementare Theorie der isolierten Anordnung*" (p.294). In footnote 1 on page 293 he asserted that his results on the theory of dense order supplemented those reported in Langford's 1927 paper. (Literally: "*Die unten angegebenen Tatsachen, die diese Theorie betreffen, bilden die Ergänzung der Ergebnisse Langfords*"). He used the phrase "*sukzessiven Elimination der Operatoren*" (successive elimination of quantifiers) in print for the first time on pp. 293 and 295… somewhat odd considering he had been promoting the method since 1926 or earlier.

[42] By "a theory of an order type α" is meant, the set of all first-order sentences true in any structure $<X, R>$, where the relation R orders the set X according to the type α.

[43] See [78]. Precise attribution of specific results is a long and complicated story. See especially the historical comments on page 1.

Again using quantifier elimination he obtained a series of decidability results on "elementary algebra and geometry" and "related systems".[44] It is worth noting a few of these results that are pertinent to the ordered field of real numbers. (For those related to geometry, see §3 above).

Let $\mathcal{R} = \langle \mathrm{R}, +, \cdot, >, 0, 1, -1 \rangle$ be this field, where $+, \cdot, >, 0, 1$ and -1 have their usual meanings. The system of elementary algebra is by definition the set $\mathrm{Th}(\mathcal{R})$ of all first-order sentences formulable in the vocabulary of $\langle \mathrm{R}, +, \cdot, >, 0, 1, -1 \rangle$ which are *true* in $\mathcal{R}$. Tarski proved that the theory $\mathrm{Th}(\mathcal{R})$ admits effective elimination of quantifiers, and he described a decision procedure for it. Moreover, he showed that $\mathrm{Th}(\mathcal{R})$ is axiomatized by the first-order axioms for real closed fields and, in consequence, any two real closed fields are arithmetically indistinguishable (or, using Tarski's later terminology, elementarily equivalent). As a by-product, one gets the following important theorem:

- *a set of real numbers is first-order definable in the field* $\mathcal{R}$ *iff it is a sum of a finite number of intervals* (*open or closed, bounded or unbounded*) *with algebraic end points.*

In [48^m](1), p.45, Tarski asked about the decidability of the elementary theory of the real field expanded by the "exponential with a fixed base, for example base 2", i.e., the theory $\mathrm{Th}(\langle \mathrm{R}, +, \cdot, >, 0, 1, -1, Exp \rangle)$, where *Exp* is the unary operation given by the formula $y = 2^x$ for all $x \in \mathrm{R}$. Commenting on this problem Tarski wrote, "The decision problem for the system just mentioned is of great theoretical and practical interest. But its solution seems to present considerable difficulties. These difficulties appear, however, to be of purely mathematical (not logical) nature: they arise from the fact that our knowledge of conditions for solvability of equations and inequalities in the enlarged system is far from adequate." In [67ma], p.38— an earlier version of [48^m], originally to have appeared in 1940, with an emphasis on completeness in its title—Tarski posed the problem of finding a complete set of axioms for the theory $\mathrm{Th}(\langle \mathrm{R}, +, \cdot, >, 0, 1, -1, Exp \rangle)$, commenting on it in a similar vein: "The attempt to carry out [this task] is confronted by difficulties of a purely mathematical nature which nevertheless do not appear to raise any question of principle".[45]

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[44] These results were announced in a variety of published works, scattered over time, and in varying notations and terminologies. See: [31], [48^m], [48^m](1), [49^ad], and [67^m]. The last was originally intended for publication in 1939 but was interrupted by the Second World War.

[45] Tarski was perfectly right that difficulties resided in mathematics, and this has been confirmed by the further development of algebraic geometry. The importance of the two problems is discussed in: D.E. Marker, *Model Theory and Exponentiation*, ***Notices of the American Mathematical Society***, vol. 43 (1996), pp. 753–759.

Tarski's theorem that the ordered field of real numbers (a first-order theory) admits quantifier elimination triggered a stream of research. Important contributions were made by: Abraham Seidenberg (1954), Abraham Robinson (1959 and 1971), Stanisław Łojasiewicz (1964–65), Paul Joseph Cohen (1969), Joseph R. Shoenfield (1971), George Edwin Collins (1982), and Helmut Wolter (1986). They are summarized in an expert presentation by Lou van den Dries: *Alfred Tarski's Elimination Theory for Real Closed Fields*, ***The Journal of Symbolic Logic***, vol. 53 (1988), pp. 7–19.

A comprehensive review by Charles I. Steinhorn of Alex J. Wilkie's paper *Model Completeness Results for Expansions of the Ordered Field of Real Numbers* in ***The Journal of Symbolic Logic***, vol.64 (1999), pp. 910–913, contains a survey of research up to the end of the 1990s stimulated by Tarski's beautiful, influential, and

Another question raised by Tarski in [48^m] was about the decidability of the theory of the structure $\langle R, +, \cdot, >, 0, 1, -1, Al \rangle$ where the relation $Al(x)$ means "x is an algebraic number". Abraham Robinson proved (1959) that one obtains a complete theory by adding to the axioms for the real closed ordered fields sentences which state that the elements of the model which satisfy $Al(x)$ constitute a real closed proper subfield which is dense in the entire model.

Tarski once opined, "In fact, I am rather inclined to agree with those who maintain that the moments of greatest creative advancement in science frequently coincide with the introduction of new notions by means of definition".[46] As Feferman and Feferman observed, a touch swellheaded. But he knew he could get away with it. Tarski himself defined many notions that contributed enormously to advances in logic, metalogic and metamathematics. A general theory of undecidability could never have been created if Tarski had not given precise mathematical definitions of two concepts: (1) that of an essentially undecidable theory, and (2) of the interpretability of one theory in another.

The two concepts were introduced, and a theory of undecidability expounded, in the paper *A General Method in Proofs of Undecidability*, which constituted Chapter 1 of the book [53^m], **Undecidable Theories**. Though authorship of the book as a whole was credited to Tarski in collaboration with Andrzej Mostowski and Raphael M. Robinson (and with some later contributions by Julia Robinson), this first part was penned by Tarski alone.

Theories considered in [53^m] were all formalized in a standard way within first-order predicate logic with identity (without predicate variables). The set of syntactically articulable formulas of the language of any theory was general recursive. A set of *valid sentences* was identified for each theory T, subject to the condition that the set be closed under consequence operations of predicate logic. T was said to be *axiomatizable* if every valid sentence could be logically derived from a fixed recursive set of axioms, and *finitely axiomatizable* if the set of axioms could be finite. If the set of valid sentences was recursive, T was said to be *decidable*; otherwise T was called *undecidable*.

A theory T_2 was said to be an *extension* of a theory T_1 (and T_1 a *subtheory* of T_2) if every valid sentence of T_1 was valid in T_2. Two theories T_1 and T_2 were said to be *compatible* if they had a common consistent extension—equivalent to saying that the union of T_1 and T_2 was consistent. And finally, a theory T was said to be *essentially undecidable*, if it was not only undecidable, but also every consistent extension of T with the same constants as T was undecidable.[47, 48] Key properties of undecidability and essential undecidability were then presented in several theorems, among them the following:

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far-reaching monograph [48^m], *A Decision Method for Elementary Algebra and Geometry*. See also Bob F. Caviness and Jeremy Russell Johnson (eds.), *Quantifier Elimination and Cylindrical Algebraic Decomposition*. Wien, New York: Springer 1998.

[46] See [44a], p. 359.

[47] Tarski's use of the word "essentially" here came from a distinction he drew between two different ways of extending a theory. An extension T_2 of T_1 was called <u>inessential</u>, if every constant of T_2 which did not occur in T_1 was an individual constant, and every valid sentence of T_2 was derivable from a set of valid sentences of T_1.

- *Let* T_1 *and* T_2 *be two compatible theories such that every constant of* T_2 *is also a constant of* T_1. *If* T_2 *is finitely axiomatizable and essentially undecidable, then* T_1 *is undecidable, and so is every subtheory of* T_1 *which has the same constants as* T_1 (Theorem 6, page18).

To exploit all these mechanisms, Tarski needed two more notions: (1) interpretability, and (2) relativization of quantifiers.

First, interpretability. Let T_1 and T_2 be two theories. Assume, first, that they have no non-logical (or "specific") constants in common. In this case, T_2 is said to be *interpretable* in T_1, if T_1 can be extended by adding to its set of valid sentences "possible" definitions of the non-logical constants of T_2 in such a way, that the resulting extension of T_1 turns out to be an extension of T_2 as well. Then, in the case when T_1 and T_2 do have some non-logical constants in common, first rewrite T_2 with new non-logical constants that do not occur in T_1, but without changing the syntactical structure of T_2 in any other respect. If the resulting theory T_2' is interpretable in T_1, then T_2 may be said to be interpretable in T_1 as well. And last, a theory T_2 is said to be *weakly interpretable* in T_1, if T_2 is interpretable in some consistent extension of T_1 with all the same constants (both logical and non-logical).

Now, relativization of quantifiers.[49] Let P be any unary predicate constant not in the language of T. For every formula Ψ of T in which P does not occur, replace all occurrences of $\forall x \Psi$ with $\forall x (Px \rightarrow \Psi)$, and replace all occurrences of $\exists x \Psi$ with $\exists x (Px \wedge \Psi)$. Call the result $T^{(P)}$. Notice that this construction creates a one-to-one mapping from the formulas of T to the formulas of $T^{(P)}$. Apply this mapping to the set of valid sentences in T, then apply standard consequence operations to the image-set to derive its logical closure in T^P, and let that closure be the set of valid sentences of $T^{(P)}$. Then $T^{(P)}$ is said to be *obtained from* T *by relativization of the quantifiers of* T *to P*.

The requirement that P initially not occur in the language of T was, as things turned out, not logically necessary. But Tarski could not prove his main meta-theorem about $T^{(P)}$ without making this assumption.

The transformation of T into $T^{(P)}$ preserves some important properties of T in $T^{(P)}$. For instance…
   (1) $T^{(P)}$ is axiomatizable iff T is axiomatizable.
   (2)   When only finitely many individual constants and operator symbols occur in T, $T^{(P)}$ is finitely axiomatizable iff T is finitely axiomatizable.

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[48] By the work of Church and Rosser from the year 1936, elementary Peano arithmetic is essentially undecidable.

[49] The general idea of relativization was introduced by Tarski in 1930. The idea of relativization of quantifiers was due to Adolf Lindenbaum and Tarski, and dated from 1935. See [56ᵐ], p.69, p.314 (footnote 1) and p.396. With Tarski as one of his doctoral advisors, Andrzej Mostowski used the method of relativization of quantifiers in his Ph.D. dissertation, which he published under the title *O niezależności definicji skończoności w systemie logiki <On the Independence of Definitions of Finiteness in a System of Logic>*, in **Dodatek do Rocznika Polskiego Towarzystwa Matematycznego** (Supplement to the **Annales de la Société Polonaise de Mathématique**), Vol. 11 (1938), pp. 1–54.

(3) $\mathsf{T}^{(P)}$ is essentially undecidable iff T is essentially undecidable (Theorems 8 & 9).

By merging interpretability with relativization one obtains *relative interpretability*. A theory T_2 is said to be *relatively interpretable* [/*relatively weakly interpretable*] in a theory T_1 iff the correlated theory $\mathsf{T}_2^{(P)}$ (obtained by relativizing the quantifiers of T_2 to a predicate P) is interpretable [/weakly interpretable] in T_1 in the senses defined above.

To apply this apparatus, one needs a simple, finitely axiomatizable, essentially undecidable theory. The second paper in [53[m]], *Undecidability and Essential Undecidability in Arithmetic*, written jointly by A. Mostowski, R.M. Robinson, and A. Tarski, constructs just such a theory. This is the famous theory Q, a subtheory of Peano arithmetic. It has seven simple axioms describing the successor function (a unary operation), multiplication and addition (binary operations), and the number zero. The authors show that Q is essentially undecidable, since any recursive function of a single variable is metamathematically definable in Q. The proof relies on Julia Robinson's characterization of recursive functions of a single variable, and a version of Tarski's theorem on undefinability of arithmetical truth, which together make it possible to avoid having to construct a special provability predicate.

In the third paper of [53[m]], *Undecidability of the Elementary Theory of Groups*, written by Tarski alone, the headline result is proved by demonstrating the following…

(i) The complete theory $\mathrm{Th}(\langle I, +, \cdot \rangle)$ of integer arithmetic is undecidable, since there exists a natural relativization of Q that is interpretable in $\mathrm{Th}(\langle I, +, \cdot \rangle)$.

(ii) The theory $\mathrm{Th}(\langle I, +, \cdot \rangle)$ is itself interpretable in the elementary theory of groups in such a way, that a decision procedure for the theory of groups would yield a decision procedure for $\mathrm{Th}(\langle I, +, \cdot \rangle)$.

Other results obtained by the method of interpretation or relative interpretation include the undecidability of the elementary theory of lattices, of modular lattices, of modular lattices with complementation, and of rings. The method was also used in proving a weak system of set theory undecidable (see [52b], by Szmielew and Tarski). In the 1960s Tarski and Szczerba used the method of interpretation to investigate the decision problem in geometry.

Tarski's talk at the 1946 Princeton *Logic Conference on Problems of Mathematics* began with a short reflexion of a semantic nature:

> Now the word "problem" has two distinct senses: in one sense, a problem is a definite question like "Is such-and-such the case?"; in another sense, we mean by a problem something of a less determined nature—which could perhaps more properly be characterized as a task—such as "Construct something with such-and-such properties." It is in this second—more general, if you prefer—sense that I will call the attention of this assembly to some important unsolved problems in mathematical logic.

Tarski's later publications treated the decision problem as a problem in the second, more general sense, recasting it as a challenge to explore what decidability is really all about, what are the different kinds of decidability, and which kinds are worth investigating in more

detail. He identified what he called *restricted* decision problems, and *second-degree* decision problems.[50]

By *restricted* decision problems Tarski meant, "problems of determining whether a set *S* of all valid sentences of a theory T *satisfying certain additional conditions* is recursive". He cited Hilbert's tenth problem, the word problem for groups, and other word problems as prime examples of problems of this type.[51] In [87^{m}], §5.5 & §8.5, he gave three more examples, and proved they were answerable in the negative:[52]

- Certain subsystems of classical propositional logic, finitely axiomatizable under the rules of substitution and detachment, are undecidable.

- The equational theory of relation algebras and the equational theory of representable relation algebras are undecidable[53].

- The equational theory of omega-relation algebras is finitely based[54] and essentially undecidable.

By *second-degree* decision problems, Tarski understood a catch-all category of meta-questions about questions—a category more easily limned by "*e.g.*" than by "$=_{def.}$". He gave his first example at the Princeton address, citing the (at that time open) question "of finding a procedure to tell whether a given set of formulas is adequate as a set of axioms for the sentential calculus".[55] In [53^{m}], pp. 34–35, he considered "the problem of the existence of a method which would permit us in each particular case to decide whether or not a given theory is decidable". Narrowing this down to something more tractable, he asked if the family of all finite sets of sentences which are axiom sets for decidable first-order theories with standard formalizations is recursive.

Based on the provable existence of an essentially undecidable theory, and on specific properties of deducibility in theories with standard formalizations, Tarski concluded that these questions were answerable in the negative. Analogous reasoning gave him a negative answer to the question: Is the family of all finite consistent sets of sentences recursive?[56]

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[50] Both terms were coined in [53^{m}] (see pp. 34–35).

[51] See [53^{m}], p. 35, and [68], p.287.

[52] To obtain these proofs he applied a method that he had earlier set out, in [53^{m}], p.22, footnote 17, which could be termed "generalized interpretation".

[53] For a class **K** of similar algebras (of a fixed similarity type τ), the equational theory of this class is defined as the set of all identities (in type τ) which are true or hold in every algebra belonging to the class **K**.

[54] In equational logic *finitely based* means axiomatizable by a finite number of equations (identities).

[55] That such a procedure does not exist was communicated in 1949 by Samuel Linial (Gulden) and Emil Leon Post. For a detailed presentation of their result, see Mary Katherine Yntema, *A detailed argument for the Post-Linial theorems*, **Notre Dame Journal of Formal Logic**, vol. 5 (1964), pp. 37–51.

[56] See: George F. McNulty, *Alfred Tarski and Undecidable Theories*, **The Journal of Symbolic Logic**, vol. 51 (1986), pp. 890–898.

Abraham A. Fraenkel, Yehoshua Bar-Hillel, and Azriel Lévy surveyed a sweeping cast of second-degree decision problems in their ***Foundations of Set Theory***. They wrote:[57]

> "The reader might have already asked himself whether there might not exist a decision method by which it could be effectively determined for every given formalized theory, whether it is decidable or not. However, a rather simple argument shows that, at any rate for finitely axiomatizable first-order theories, no such general method could possibly exist, hence that this second-degree decision problem, so to speak, is unsolvable. [The authors cite [53[m]], p.35]. Other higher-degree decision problems deal, for instance, with the existence of an effective procedure of deciding, for every given *presentation* of a certain algebraic structure, such as of a semi-group, of a semi-group with cancellation, or of a group, whether the structures defined by these presentations have certain algebraic properties such as cyclicity, finiteness, simplicity, decomposability into a finite product, etc. … Aspects of relative decidability problems were treated by Post [1944] who introduced the term "*degree of recursive unsolvability ...*""

In [68], on page 287, Tarski wrote, "Various notions, problems, and results discussed so far in this paper suggest in a natural way corresponding *decision problems*. These are problems of the type: is a given set of equations, or of finite sets of equations, or of finite algebras, recursive? The meaning of the term *recursive* in these contexts is clear; finite algebras can be regarded as algebras whose universe consists of finitely many integers."

He set out examples of such decision problems on pp. 287–288. Let Σ range over finite subsets of equations of a fixed finite similarity type—i.e., equations in which only a finite number of operation symbols occur. Then the following six condition-schemata, with free variable Σ, can be postulated:

c1. Σ is a base for a given finite algebra $\boldsymbol{a}$.

c2. There is a finite algebra $\boldsymbol{a}$ for which Σ is a basis.

c3. For a given positive integer k, the equational theory generated by Σ has an independent base consisting of k equations.

c4. The equational theory generated by Σ is consistent.

c5. The equational theory generated by Σ is complete.

c6. The equational theory generated by Σ is decidable.

To each of the above six condition-schemata (c_i) there corresponds the decision problem (P_i): *Is the family of all Σ such that Σ has property* (c_i) *recursive*? Six parallel decision problems can be obtained when Σ is restricted to range over one-element sets of equations.

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[57] A. Fraenkel, Y. Bar-Hillel, A. Lévy, ***Foundations of Set Theory***, **Studies in Logic and the Foundations of Mathematics**, Vol. 67, Elsevier, Amsterdam 1958, (2nd edition 1973). The quotation is from the second edition, Chapter 5, §7, *The limitative theorems of Gödel, Tarski, Church and their generalizations*, page 320.

Also consider the following three condition-schemata on finite algebras, with free variable a, and their related decision problems:

e1. a is finitely based (that is, the set of all equations true in a is axiomatizable by a finite set of equations).

e2. a is equationally complete.

e3. a has an independent base consisting of k elements, where $k \leq \aleph_0$.

Of these, the problem associated with condition (e1), known as *Tarski's finite basis problem*, turned out to be most difficult, and attempts to solve it profoundly influenced the development of universal algebra and computability theory for thirty years. It was only in 1996 that Ralph McKenzie announced the result that *the class of all finite algebras which are finitely based is not recursive*.[58]

Flashback and flashforward can be revealing devices not just in cinematography. Here is Tarski speaking in 1968: [59]

> "There is much affinity between the formalisms of sentential calculus and equational logic; as a consequence various metalogical results established for sentential calculus can frequently be carried over to equational logic with appropriate changes in formulations and proofs."

Here are Łukasiewicz and Tarski speaking in 1930: [60]

> "In conclusion we should like to add that, as the simplest deductive discipline, the sentential calculus is particularly suitable for metamathematical investigations. It is to be regarded as a laboratory in which metamathematical methods can be discovered and metamathematical concepts constructed which can then be carried over to more complicated mathematical systems."

And here, through the medium of Steven Givant, is Tarski speaking from beyond the grave in 1987: [61]

> "The above observations may seem somewhat paradoxical. The formalism $\mathcal{T}$ of two-valued sentential logic is usually regarded as the simplest and most trivial logical formalism, with an almost empty mathematical content. Nevertheless, the formalism $\mathcal{L}_r^\times$, so closely related to $\mathcal{T}$ in its syntactical part, presents an adequate basis for the development of set theory, which is, in a sense, the richest mathematical discipline; and even in its logical part $\mathcal{L}_r^\times$ embodies an interesting and far from trivial mathematical theory, namely the equational theory of relation algebras.
>
> "One conclusion emerges from our discussion: the connection between the formal structure of the language and its intended semantical interpretation is much looser than we might be inclined to believe.

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[58] R. McKenzie, *Tarski's Finite Basis Problem is Undecidable*, **International Journal of Algebra and Computation**, Vol. 6 (1996), pp. 49–104.

[59] See [68], p. 288.

[60] See [56^m], p. 59.

[61] See [87^m], pp. 167–168.

"It may be interesting to observe that the logic of   is well known to be decidable, while the logic of $\mathcal{L}_r^\times$ is undecidable… Thus, we have obtained an example of an undecidable subtheory of the two-valued sentential logic (in fact a subtheory based upon a finite set of axiom schemata). Also ... this subtheory can be supplemented by means of finitely many axioms to form an essentially undecidable theory $\boldsymbol{\theta}$; this will be a theory in the same formalism (i.e., $\mathcal{L}_r^\times$) as sentential logic, but clearly not a subtheory of that logic."

§5. Selected Works of Tarski

Listed below are all of Tarski's works cited or referred to in §§1–4 above. Also listed are a few other works which, while not cited in §§1–4, are central to his canon. The method and style of citing Tarski's works adhere as closely as possible to the conventions adopted in Steven R. Givant's *Bibliography of Alfred Tarski*, in **The Journal of Symbolic Logic**, Vol. 51, No. 4 (Dec., 1986), pp. 913–941.

Givant's bibliography was updated, corrected, and translated into Polish by Jan Zygmunt in [95[m]], pp. 333–372. Both were divided into the same ten sections: Papers, Abstracts, Monographs, Exercises and problems, Contributions to discussions, Reviews, Publication as editor, Project reports, Letters, and Appendix. Both give full information on later re-editions and translations of an item, as well as references to reviews in **Mathematical Reviews** and **The Journal of Symbolic Logic**. The bibliography below is divided into only four sections: (A) Monographs, (B) Papers, (C) Abstracts, and (D) Publications as editor.

A more recently updated bibliography has been supplied by Andrew McFarland, Joanna McFarland and James T. Smith in [14[m]]. See that volume's Ch. 16: *Posthumous Publications*, Ch. 17: *Biographical Studies*, Ch. 18: *Research surveys*, and especially its excellent *Bibliography*.

A. Monographs

[33[m]] *Pojęcie prawdy w językach nauk dedukcyjnych*. **Prace Towarzystwa Naukowego Warszawskiego, Wydział III—Nauk Matematyczno-fizycznych**, nr. 34, Warszawa 1933, vii + 116 pp. + errata. (see [35b] for German translation.)

[36[m]] *O logice matematycznej i metodzie dedukcyjnej*. **Bibljoteczka Matematyczna**, vol. 3–5, Książnica-Atlas, Lwów and Warszawa 1936, 167 pp.
(1) *Einfürung in die mathematiche Logik und in die Methodologie der Mathematik* Julius Springer Verlag, Vienna, 1937, x + 166 pp. (German translation of [36[m]].)

[41[m]] *Introduction to Logic and to the Methodology of Deductive Sciences*. Oxford University Press, Oxford and New York 1941, xviii + 239 pp. [Fourth, enlarged edition of [41[m]], edited by Jan Tarski, **Oxford Logic Guides**, vol. 24, Oxford University Press, New York – Oxford 1994, xxiv + 229 pp.]

[48^m] *A Decision Method for Elementary Algebra and Geometry* (prepared for publication by J.C.C. McKinsey) U.S. Air Force Project RAND, R-109, the RAND Corporation, Santa Monica, California 1948, iv + 60 pp.
(1) Second, revised edition of [48^m] (prepared for publication with the assistance of J.C.C. McKinsey), University of California Press, Berkeley and Los Angeles, California 1951, iii + 63 pp.

[49^m] *Cardinal Algebras, With an Appendix: Cardinal Products of Isomorphism Types* (by B. Jónsson and A. Tarski), Oxford University Press, Oxford and New York 1949, xii + 327 pp.

[53^m] *Undecidable Theories* (with A. Mostowsk and R. M. Robinson), North-Holland Publishing Company, Amsterdam 1953, xii + 98 pp.

[56^m] *Logic*, *Semantics*, *Metamathematics*. *Papers from 1923 to 1938* (translated by J.H. Woodger), Clarendon Press, Oxford 1956, xiv + 471 pp. [English translation of [23].]
(1) Second, revised edition, with editor's introduction and an analytic index (J. Corcoran, editor), Hackett Publishing Company, Indianapolis, Indiana 1983, xxx + 506 pp.

[56ma] *Ordinal Algebras*, with appendices: *Some Additional Theorems on Ordinal Algebras* (by C.C. Chang) and *A Unique Decomposition Theorem for Relational Addition* (by B. Jónsson), North-Holland Publishing Co., Amsterdam 1956, 133 pp.

[67ma] *The completeness of elementary algebra and geometry*, Institut Blaise Pascal, Paris 1967, iv + 50 pp. (A reprint from page proofs of the work scheduled to appear in 1940 in **Actualités Scientifiques et Industrielles**, Hermann & Cie, Paris. See [48^m].)

[71^m] *Cylindric Algebras*. *Part I*. *With an Introductory Chapter*: *General Theory of Algebras* (with L. Henkin and J.D. Monk), North-Holland Publishing Company, Amsterdam 1971, vi + 508 pp.

[83^m] *Metamathematische Methoden in der Geometrie* (with W. Schwabhäuser and W. Szmielew), Springer-Verlag, Berlin 1983, viii + 482 pp.

[85^m] *Cylindric Algebras. Part II* (with L. Henkin and J.D. Monk), North-Holland, Amsterdam 1985, vii + 302 pp.

[86^m] *The Collected Works of Alfred Tarski*, vol. 1, *1921–1934*; vol. 2, *1935–1944*; vol. 3, *1945–1957*; vol. 4, *1958–1979* (S.R. Givant and R.N. McKenzie, editors), Birkhäuser, Basel, Boston and Stuttgart 1986, xii + 658 pp. (vol. 1); xii + 699 pp. (vol. 2); xii + 682 pp. (vol. 3); xii + 757 pp. (vol. 4). [Volume 4 (pp. 739–757) contains a reprint of S. Givant, *Bibliography of Alfred Tarski*, **JSL**, vol. 45 (1986), pp.913–941]

[87^m] ***A Formalization of Set Theory without Variables***. (with S.R. Givant), **Colloquium Publications**, vol. 41, American Mathematical Society, Providence, Rhode Island 1987, xxi + 318 pp.

[95^m] ***Pisma logiczno-filozoficzne***, vol. 1: ***Prawda***. (Translated and annotated, with an introduction by Jan Zygmunt), **Biblioteka Współczesnych Filozofów**, Wydawnictwo Naukowe PWN, Warszawa 1995, xxiv + 390 pp. + errata.

[01^m] ***Pisma logiczno-filozoficzne***, vol. 2: ***Metalogika***. (Translated and annotated, with an intoduction by Jan Zygmunt), **Biblioteka Współczesnych Filozofów**, Wydawnictwo Naukowe PWN, Warszawa 2001, xiv + 516 pp.

[14^m] ***Alfred Tarski***: ***Early Work in Poland—Geometry and Teaching***. ***With a Bibliographic Supplement***. (A. McFarland, J. McFarland, James T. Smith, editors), Birkhäuser: New York 2014, xxiii + 499 pp.

B. Papers

[21] *Przyczynek do aksjomatyki zbioru dobrze uporządkowanego <A Contribution to the Axiomatics of Well-Ordered Sets>*, **Przegląd Filozoficzny**, vol. 24 (1921), pp. 85–94.

[24a] *Sur quelques théorèmes qui équivalent à l'axiome du choix*, **Fundamenta Mathematicae**, vol. 5 (1924), pp. 147–154.

[24b] *O równoważności wielokątów <On the Equivalence of Polygons>*, **Przegląd Matematyczno-fizyczny**, vol. 2 (1924), pp. 47–60.

[24c] *Sur les ensembles finis.* **Fundamenta Mathematicae**, vol. 6 (1924), pp. 45–95.

[24d] *Sur la décomposition des ensembles de points en parties respectivement congruentes* (with S. Banach), **Fundamenta Mathematicae**, vol. 6 (1924), pp. 244–277.

[25] *Quelques théorèmes sur les alephs*, **Fundamenta Mathematicae**, vol. 7 (1925), pp. 1–14.

[26] *Communication sur les recherches de la théorie des ensembles* (with A. Lindenbaum), **Sprawozdania z Posiedzeń Towarzystwa Naukowego Warszawskiego**, **Wydział III Nauk Matematycznych i Przyrodniczych** (= **Comptes Rendus des Séances de la Société des Sciences et des Lettres de Varsovie, Classe III**), vol. 19 (1926), pp. 299–330.

[29a] *Geschichtliche Entwicklung und gegenwärtiger Zustand der Gleichmächtigkeits-theorie und der Kardinalzahlarithmetik*, **Księga Pamiątkowa Pierwszego Polskiego Zjazdu Matematycznego** (Supplement to **Rocznik Polskiego Towarzystwa Matematycznego**), Kraków, 1929, pp. 48–54.

[29b]   *Sur les fonctions additives dans les classes abstraites et leur application au problème de la mesure. **Sprawozdania z Posiedzeń Towarzystwa Naukowego Warszawskiego**, **Wydział III Nauk Matematycznych i Przyrodniczych (= Comptes Rendus des Séances de la Société des Sciences et des Lettres de Varsovie, Classe III**), vol. 22 (1929 – published 1930), pp. 114–117.

[30]   *Une contribution à la théorie de la mesure*, **Fundamenta Mathematicae**, vol. 15 (1930), pp. 42–50.

[30a]   *Sur une propriété caractéristique des nombres inaccessibles* (with W. Sierpiński), **Fundamenta Mathematicae**, vol. 5 (1930), pp. 292–300.

[30b]   *Über Äquivalenz der Mengen in Bezug auf eine beliebige Klasse von Abbildungen*, **Atti del Congresso Internazionale dei Matematici**, **Bologna, 3–10 settembre 1928**, vol. 6, Nicola Zanichelli, Bologna 1930, pp. 243–252.

[30c]   *Über einige fundamentalen Begriffe der Metamathematik*, **Sprawozdania z Posiedzeń Towarzystwa Naukowego Warszawskiego**, **Wydział III Nauk Matematyczno-fizycznych** (= **Comptes Rendus des Séances de la Société des Sciences et des Lettres de Varsovie, Classe III**), vol. 23 (1930), pp. 22–29.

[30d]   *Untersuchungen über den Aussagenkalkül* (with J. Łukasiewicz), **Sprawozdania z Posiedzeń Towarzystwa Naukowego Warszawskiego**, **Wydział III Nauk Matematyczno-fizycznych** (= **Comptes Rendus des Séances de la Société des Sciences et des Lettres de Varsovie, Classe III**), vol. 23 (1930), pp. 30–50.

[30e]   *Fundamentale Begriffe der Methodologie der deduktiven Wissenschaften. I*, **Monatshefte für Mathematik und Physik**, vol. 37 (1930), pp. 361–404.

[30f]   *Sur les classes d'ensembles closes par rapport à certaines opérations élémentaires*, **Fundamenta Mathematicae**, vol. 16 (1930), pp. 181–304.

[31]   *Sur les ensembles définissables de nombres réels. I*, **Fundamenta Mathematicae**, vol. 17 (1931), pp. 210–239.

[31a]   *Les opérations logiques et les ensembles projectifs* (with C. Kuratowski), **Fundamenta Mathematicae**, vol. 17 (1931), pp. 240–248.

[31b]   *O stopniu rownoważnosci wielokątów <On the Degree of Equivalence of Polygons>*, **Młody Matematyk**, vol. 1 (Supplement to **Parametr**, vol. 2) (1931), pp. 37–44.

[34]   *Z badań metodologicznych nad definjowalnością terminów <Methodological Investigations into the Definability of Terms>*, **Przegląd Filozoficzny**, vol. 37 (1934), pp. 438–460.

[35a]   *Grundzüge des Systemenkalküls. Erster Teil*, **Fundamenta Mathematicae**, vol. 25 (1935), pp. 503–526.

[35b]   *Der Wahrheitsbegriff in den formalisierten Sprachen*, **Studia Philosophica**, vol. 1 (1935), pp. 261–405. (German translation of [33[m]] by L. Blaustein, with an appendix. Also printed separately in Lwów, 1935, 145 pp.)

[36]   *O ugruntowaniu naukowej semantyki <On Foundations for a Semantics of Science>*, **Przeglqd Filozoficzny**, vol. 39 (1936), pp. 50–57.

[36a]   *O pojęciu wynikania logicznego <On the Concept of Logical Consequence>*, **Przeglqd Filozoficzny**, vol. 39 (1936), pp. 58–68.

[36b]   *Über die Beschränktheit der Ausdrucksmittel deduktiver Theorien* (with A. Lindenbaum), **Ergebnisse eines Mathematischen Kolloquiums**, vol. 7 (1936), pp. 15–22.

[36d]   *Grundzüge des Systemenkalküls. Zweiter Teil*, **Fundamenta Mathematicae**, vol. 26 (1936), pp. 283–301.

[38a]   *Über unerreichbare Kardinalzahlen*, **Fundamenta Mathematicae**, vol. 30 (1938), pp. 68–89.

[38c]   *Ein Überdeckungsästz für endliche Mengen*, **Fundamenta Mathematicae**, vol. 30 (1938), pp. 156–163.

[38d]   *Eine äquivalente Formulierung des Auswahlaxioms*, **Fundamenta Mathematicae**, vol. 30 (1938), pp. 197–201.

[38g]   *Algebraische Fassung des Maßproblems*, **Fundamenta Mathematicae**, vol. 31 (1938), pp. 47–66.

[38h]   *Der Aussagenkalkül und die Topologie*, **Fundamenta Mathematicae**, vol. 31 (1938), pp. 103–134.

[39]   *Ideale in vollständigen Mengenkörpern. I*, **Fundamenta Mathematicae**, vol. 32 (1939), pp. 45–63.

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[52b]   *Mutual interpretability of some essentially undecidable theories* (with W. Szmielew), ***Proceedings of the International Congress of Mathematicians. Cambridge, Massachusetts, U.S.A., August 30 –September 6, 1950***, vol. 1, American Mathematical Society, Providence, Rhode Island 1952, p. 734.

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[56b]   *Equilaterality as the only primitive notion of Euclidean geometry* (with E.W. Beth), ***Indagationes Mathematicae***, vol. 18 (1956), pp. 462–467.

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[59]   *What is elementary geometry?* in [59[e]], pp. 16–29.

[61a]   *Cylindric algebras* (with L. Henkin), ***Lattice theory***, **Proceedings of Symposia in Pure Mathematics**, vol. 2 (R.P. Dilworth, editor), American Mathematical Society, Providence, Rhode Island, 1961, pp. 83–113.

[61b]   *On some problems involving inaccessible cardinals* (with Paul Erdös), ***Essays on the Foundations of Mathematics*** (Y. Bar-Hillel, E.I.J. Poznański, M.O. Rabin and A. Robinson, editors), Magnus Press, Jerusalem 1961, pp. 50–82.

[62]   *Some problems and results relevant to the foundations of set theory*, in [62[e]], pp. 125–135.

[64]   *From accessible to inaccessible cardinals. Results holding for all accessible cardinal numbers and the problem of their extension to inaccessible ones* (with H.J. Keisler), **Fundamenta Mathematicae**, vol. 53 (1964), pp. 225–308.

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[65a] *Metamathematical properties of some affine geometries* (with L.W. Szczerba), **Logic, Methodology and Philosophy of Science. Proceedings of the 1964 International Congress** (Y. Bar-Hillel, editor), North-Holland Publishing Company, Amsterdam 1965, pp. 166–178.

[68]   *Equational logic and equational theories of algebras*, **Contributions to Mathematical Logic. Proceedings of the Logic Colloquium, Hannover 1966** (H.A. Schmidt, K. Schütte and H.J. Thiele, editors), North-Holland Publishing Company, Amsterdam 1968, pp. 275–288.

[78]   *The elementary theory of well-ordering—A metamathematical study* (with J. Doner and A. Mostowski), **Logic Colloquium '77** (A. MacIntyre, L. Pacholski and J. Paris, editors), North-Holland Publishing Company, Amsterdam 1978, pp. 1–54.

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